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On Projective R_f - modules over Polynomial Rings

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In this paper we get an affirmative answer of the question of Quillen by Nashier's method. This paper is devided into two sections. In § 1, we recall a few definitions and results for § 2 where we prove our main theorems.

1. Preliminaries

The Bass-Quillen conjecture first was posed as a problem in [1, H. Bass, Problem IX (1972)] and then as a conjecture in [15, Quillen, 1976]. The possibility of reducing the case of deimension $\leqq$ d to the local case is essentially due to Quillen.

Bass-Quillen conjecture ; Let R be a commutative regular ring. Is every finitely generated projective module over $R[T]$ extended from R ?

Quillen posed the following question in a view of Quillen Patching Theorem which is a local-global criterion for a module M over polynomial rings to be extended.

Definition 1. Let R be a commutative ring. An $R[T]$ - module M will be called extended from R if there is an R module N such that $M \simeq N[T] = R[T] \otimes_R N$. In this case $N \simeq M/TM$.

Quillen's Patching Theorem ; Let M be a finitely presented module over $A = R[T]$. If M_m is extended from R_m for all maximal ideal m of R (where $M_m = A_m \otimes_A M$ and $A_m = R_m[T]$) then M is extended from R.

Question ; Let R be a regular local ring and f a regular parameter of R. Are all finitely generated projective R_f - modules free?

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When the Quillen's question has s positive solution, the Bass-Quillen conjecture also has a positive solution. But It is not clear whether a positive solution to the Bass-Quillen conjecture implies the question of Quillen in the previous studies.

Throughout all rings will be commutative Noetherian with identity element and all modules will be finitely generated. We collect some definitions and results for later use.

Definition 2. Let $f: B \to A$ be a homomorphism of rings and let h an element of B such that

(i) h is a nonzero-divisor in B

(ii) $f(h)$ is a nonzero-divisor in A.

(iii) f induces an isomorphism $B / hB \simeq A / f(h)A$.

The commutative diagram

$$\begin{array}{ccc} B & \longrightarrow & B_h \\ f \big\downarrow & & \big\downarrow f_h \\ A & \longrightarrow & A_h \end{array} \qquad (1)$$

resulting from a situation as above will be called a patching diagram. It is easy to see that B is the fibre product of B_h and A over A_h. We shall describe (1) as $f: B \to A$ is an analytic isomorphism along h or equivalently $A = B + hA$ with $hA \cap B = hB$.

Lemma 1. Let (R, m) be a local ring. A monic polynomial $f \in R[T]$ is called a Weierstrass polynomial if $f(T) = T^n + a_1 T^{n-1} + \cdots + a_n$, $a_i \in m$, for $i = 1, 2, \cdots$, n. Then we have a patching diagram

$$\begin{array}{ccc} R[T] & \longrightarrow & R[T]_{(m,\, T)} \\ R[T]_f & \longrightarrow & R[T]_{(m,\, T)}\,[1/f] \end{array}$$

Proof. Let M be a maximal ideal in $R[T]$ that contains f. Since f is monic polynomial, $M \cap R = m$. Thus $T \in M$ and hence $M = (m, T)$. Since $S = R[T]-(m, T)$ in $R[T]$ contains nonzero divisors, we have an inclusion map $R[T] \to R[T]_{(m,\, T)}$. Since f is monic, $R[T]/(f(T))$ is semilocal, and any maximal ideal n of it lies over m. But since $f(T) \in n$ and f is a Weierstrass polynomial, we have $T \in n$. Therefore, $n = (m, T)$. Thus $R[T]/(f(T))$ is local and so $R[T]/(f(T)) = R[T]/(f(T))_{(m, T)} = R[T]_{(m, T)}/(f(T))$. Therefore $R[T]_f \to R[T]_{(m, T)}[1/f]$ is an analytic isomorphism along f.

Next Lemma and Theorem, which are due to Nashier [14] and Roitman[16], plays the crucial role in the main theorem.

Lemma 2. Let $f : B \to A$ be an analytic isomorphism along h in B. Let M be a projective A-module such that $M_h \simeq A_h \otimes_{B_h} N$, where N is a projective B_h-module. Then there is a projective B-module P such that $M \simeq P \otimes_B A$ and $P_h \simeq N$.

Theorem A. Let R be a Noetherian ring. The following properties are equivalent ;

a) Finitely generated projective $R[X_1, \cdots, X_n]$-modules are extended.

b) Finitely generated projective $R_m[X_1, \cdots, X_n]$-modules of rank $\leqq$ Krull $dim\ R_m$ are free for each maximal ideal m of R.

c) Finitely generated projective $R[X_1, \cdots, X_n]$-modules of rank $\leqq$ Krull $dim\ R$ are extended.

We will also use the following theorem which is essentially due to Lindel[10] as a tool for the proof of the main theorem.

Theorem B. Let (R, m) be a regular local ring of dimension d over a field k. Assume that the residue filed R/m is a finite separable extension of k. Let $\{a, X_2, \cdots, X_d\}$ be a system of parameters of R such that $\{X_2, \cdots, X_d\}$ is part of a regular system of parameters of R. Then there is a regular local subring S of R with the following properties ;

(1) S is a localization of the polynomial ring $C = k[X_1, \cdots, X_d]$ at a maximal ideal of the type $(f(X_1), X_2, \cdots, X_d)$ where $f(X_1)$ is a monic irreducible polynomial in $k[X_1]$. Moreover, we may assume $X_i = f_i$ for $i = 2, \cdots, d$.

(2) There is an element h in $S \cap aR$ such that the inclusion $S \to R$ is an analytic isomorphism along h.

2. Main Theorems

In this section we prove the main theorems.

Theorem 1. Let $(D, (\pi))$ be a discrete valuation ring with infinite residue field. Let (R, m) denote the regular local ring $D[X_2, \cdots, X_d]_N$ with dimension d, where N is the maximal ideals

$(\pi, X_2, \cdots, X_d)$ of the polynomial ring $B = D[X_2, \cdots, X_d]$. Set $A = R[T]$, where $T = 1/X_2$. Then

(1) For a maximal ideal M in A, $\mu(M) = ht(M)$.

(2) A projective A-module P is free.

Proof. If $dim\ R \leqq 2$, (i) and (ii) are trivially true. So we may assume that $dim\ R = d \geqq 3$. Let M be a maximal ideal of $A = R[X_2^{-1}]$. Then $ht[M] = d-1 \geqq 2$.

Let $P = M \cap R$, where $R = D[X_2, \cdots, X_d]_N$. Let us set $Q = P \cap B$, where $B = D[X_2, \cdots, X_d]$. Since P is a prime ideal of height $\geqq 2$ in R, Q is a prime ideal in B with height $\geqq 2$. We consider the following two cases.

Case (i) $\pi \notin Q$. Since $D/(\pi)$ is an infinite field, Q contains a homogeneous polynomial f of degree $t > 0$ in B such that f has unit values in D. Let $R_1 = D[X_2, \cdots, X_{d-1}]_p[X_d]$, where $p = [\pi, X_2, \cdots, X_{d-1}]$. Since $(p\ , X_d)$ is the only maximal ideal of R_1 that contains the monic f we have that the inculsion $R_1 \to R$ is an analytic isomorphism along f. Then $R_1[1/X_2] \to R[1/X_2]$ is an analytic isomorphism along f.

We observe that the ring $A_1 = R_1[1/X_2]$ is a polynomial ring in X_d over a regular ring $D[X_2, \cdots, X_{d-1}]_p[1/X_2]$ which is a Hilbert domain. Since $f \in M \cap A_1$, $M_1 = M \cap A_1$ is a maximal ideal of A_1 such that $M_1R = M$ and $ht(M_1) = ht(M)$, So $\mu(M_1) = \mu(M_1/M_1^2) = ht(M_1)$. Thus we get $\mu(M) = \mu(M/M^2) = ht(M)$ by Prop. 2. 4. of [13]. Since Q contains a homogeneous polynomial f such that f has a unit values in D, there is a projective A - module of rank equal to $\mu(M/M^2)$ mapping onto M. So a projective A-module P is free.

Case (ii) $\pi \in P = M \cap R$. Let $\overline{M} = M\ (\pi)$. Then $\overline{M}$ is a maximal ideal in $(D/(\pi))\ [X_2, \cdots, X_d](X_2, \cdots, X_d)\ [1/X_2]$. Note that $ht(\overline{M}) = ht(M) - 1 \geqq 2$. We can follow the same argument as that given in case (i), where D will be replaced by the field $D/(\pi)$, to concluded that $\mu(\overline{M}) = ht(\overline{M})$. So $\mu(M) = \mu(M/M^2) = ht(M)$. Thus the proofs of (1) and (2) are completed.

Theorem 2. $(D, (\pi))$ be a discrete valuation ring such that residue field $D/(\pi)$ is infinite. Let $R = B_N$, where N is the maximal ideal $(\pi, X_2, \cdots, X_d)$ of the polynomial $B = D[X_2, \cdots, X_d]$. Let $f \in M - M^2$, where M is the maximal ideal of R. Then all finitely generated projective R_f-modules are free.

Proof. Let P be a projective R_f - module. As $f \in M - M^2$ we may assume that f is one of the X_i's. Say $f = X_d$. Now we proceed by induction on d. If $d = 1$, then R_f is a field and we are done. So we assume $d \geqq 2$. Let g be an element of R such that P_g is free $(R_f)_g$ - module.

Without loss of generality we may assume that g and f have no common factors in R. Hence (g, f) is a sequence in R. By Theorem B, we find h in $gR \cap R'$, where $R'=D[X_2, \cdots, X_d]_p$ and $p = (\pi, X_2, \cdots, X_d)$. Moreover $X_d = f$. Therefore $R'_{X_d} \rightarrow R_f$ is analytic isomorphism along h and P_h is free $(R_f)_h$ - module. Hence by patching diagram (1), there exists a projective R'_{X_d} - module Q such that

$$P \simeq Q \otimes_{R'_{X_d}} R_f$$

Therefore it is enough to prove that Q is free.Let $S = D[X_2, \cdots, X_{d-1}]_{(\pi, X_2, \cdots, X_{d-1})}$ and $T = X_d^{-1}$. Then $R'_{X_d} = S(T)$. Let $R_1 = S[X_d] = D[X_2, \cdots, X_{d-1}]_{(\pi, X_2, \cdots, X_{d-1})}[X_d]$. Since (p, X_d), where $p = (\pi, X_2, \cdots, X_{d-1})$, is the only maximal ideal of R_1 that contains the monic f. Then we have the patching diagram

$$\begin{array}{ccc} S[X_d] & \longrightarrow & S[X_d]_{(p, X_d)} \\ \downarrow & & \downarrow \\ S[X_d, X_d^{-1}] & \longrightarrow & S[X_d]_{(p, X_d)}[T] \end{array}$$

i. e.,

$$\begin{array}{ccc} R_1 = S[X_d] & \longrightarrow & S[X_d]_{(p, X_d)} = R' \\ \downarrow & & \downarrow \\ (R_1)_f = S[X_d]_f & \longrightarrow & S(T) = R'_{X_d} \end{array}$$

We have that $R_1 \rightarrow R'$ is an analytic isomorphism along f. Then $(R_1)_f \rightarrow R'_{X_d}$ is an analytic isomorphism along f. As Q_f is extended from $(R_1)_f$, Lemma 2 gives that Q is extended from a projective R_1 - module, say Q'. By Theorem A, we see that projective R_1 - modules are extended from S. By induction hypothesis, projective S-module is free. Hence Q is free. This completes the proof.

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