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A Note on the Coincidence Theory

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I. Introduction

Let f and g be the continuous functions from a topological space X to a topological space Y. A point x of X satisfying $f(x)=g(x)$ is called a coincidence of f and g. The set of coincidences of f and g is denoted by $\Gamma(f, g)$, that is,

$$\Gamma(f, g)=\{x\in X \mid f(x)=g(x)\}.$$

The object of coincidence theory is to provide a method for computing the number of coincidences. In [1], Brooks defined the Nielsen number which is a lower bound for the number of coincidences. Let F be a homotopy from f to f' and G be a homotopy from g to g'. The main tools are the f, g-equivalence relation on $\Gamma(f, g)$ and F, G-related relation between the coincidences of f, g and those of f', g'.

In this paper, first of all, we define a coincidence class using covering space theory as the way Jiang have done in [4] to define a fixed point class in fixed point theory ([4]. pp.4~6) and we prove that a nonepmty coincidence class is equal to a class by the f, g-equivalence relation on $\Gamma(f, g)$ (Theorem 2.7).

Next, we show that the number of coincidence classes are homotopy invariant (Theorem 2.9).

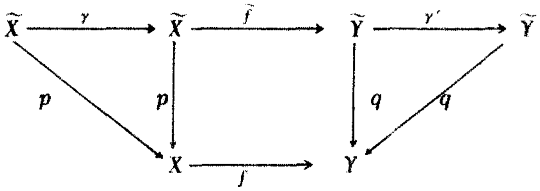
Lastly, we derive a definition (Definition 2.10) which is also obtained from a result of Jiang ([4]. p.8) and prove that it is equivalent to the F, G-related relation (Theorem 2.12).

II. Main Theorems

We always assume X and Y to be connected compact polyhedra. It is well known that a connected compact polyhedra has a universal covering space. Let $p : \widetilde{X} \to X$ and $q : \widetilde{Y} \to Y$ be the universal coverings of X and Y, respectively.

Definition 2.1. A lifting of a map $f: X \to Y$ is a map $\tilde{f}: \tilde{X} \to \tilde{Y}$ such that $q \circ \tilde{f} = f \circ p$. A covering transformation of $\tilde{X}$ is a homeomorphism $\gamma: \tilde{X} \to \tilde{X}$ such that $p \circ \gamma = p$.

The covering transformations of $\tilde{X}$ form a group, we shall denoted it by $\mathcal{D} = \mathcal{D}(\tilde{X}, p)$ which is isomorphic to the fundamental group $\Pi_1(X)$ of X. We denote the covering transformation group of $\tilde{Y}$ by $\mathcal{D}' = \mathcal{D}'(Y, q)$.



Proposition 2.2. (i) For any $x_0 \in X$ and $\tilde{x_0}, \tilde{x_0}' \in p^{-1}(x_0)$, there is a unique covering transformation $\gamma: \tilde{X} \to \tilde{X}$ such that $\gamma(\tilde{x_0}) = \tilde{x_0}'$.

(ii) Let $f: X \to Y$ be a map. For given $x_0 \in X$ and $f(x_0) = y_0$, pick $\tilde{x_0} \in p^{-1}(x_0)$ and $\tilde{y_0} \in q^{-1}(y_0)$ arbitrarily, then there is a unique lifting $\tilde{f}$ of f such that $\tilde{f}(\tilde{x_0}) = \tilde{y_0}$.

(iii) Suppose $\tilde{f}$ is a lifting of f, $\gamma \in \mathcal{D}$ and $\gamma' \in \mathcal{D}'$. Then $\gamma' \circ \tilde{f} \circ \gamma$ is a lifting of f.

(iv) For any two liftings $\tilde{f}$ and $\tilde{f}'$, there is a unique element $\gamma' \in \mathcal{D}'$ such that $\tilde{f}' = \gamma' \circ \tilde{f}$.

Proof. (i), (ii) are standard theorems in covering space theory. (iii) follows from Definition 2.1.

(iv) For any $x \in X$ and $\tilde{x} \in p^{-1}(x)$, we have $\tilde{f}(\tilde{x}), \tilde{f}'(\tilde{x}) \in q^{-1}(f(x))$.
By (i) there is a unique element $\gamma' \in \mathcal{D}'$ such that $\gamma'(\tilde{f}(x)) = \tilde{f}'(\tilde{x})$. Thus we have $\gamma' \circ \tilde{f} = \tilde{f}'$ by uniqueness of (ii).

Lemma 2.3. Suppose $\tilde{x} \in p^{-1}(x)$ is a coincidence of liftings $\tilde{f}$ and $\tilde{g}$ of f and g, respectively, and let $\gamma \in \mathcal{D}$. Then $\gamma(\tilde{x}) \in p^{-1}(x)$ is a coincidence of a lifting $\tilde{f}'$ of f and a lifting $\tilde{g}'$ of g iff there is a unique element $\gamma' \in \mathcal{D}'$ such that $\tilde{f} = \gamma' \circ \tilde{f}' \circ \gamma$ and $\tilde{g} = \gamma' \circ \tilde{g}' \circ \gamma$.

Proof. "Only if" : By (iv), there are unique elements $\gamma', \gamma'' \in \mathcal{D}'$ such that $\tilde{f} = \gamma' \circ \tilde{f}' \circ \gamma$ and $\tilde{g} = \gamma'' \circ \tilde{g}' \circ \gamma$. Then we have $\gamma' \circ \tilde{f}' \circ \gamma(\tilde{x}) = \tilde{f}(\tilde{x}) = \tilde{g}(\tilde{x}) = \gamma'' \circ \tilde{g}' \circ \gamma(\tilde{x})$
$= \gamma'' \circ \tilde{f}' \circ \gamma(\tilde{x})$ so that $\gamma' = \gamma''$.

"If". is obvious : $\gamma' \circ \tilde{f}' \circ \gamma(\tilde{x}) = \tilde{f}(\tilde{x}) = \tilde{g}(\tilde{x}) = \gamma' \circ \tilde{g}' \circ \gamma(\tilde{x})$.

Hence $\tilde{f}' \circ \gamma(\tilde{x}) = \tilde{g}' \circ \gamma(\tilde{x})$.

Definition 2.4. Two lifting pairs $(\tilde{f}, \tilde{g})$ and $(\tilde{f}', \tilde{g}')$ of f, g are said to be conjugate if there exist $\gamma \in \mathcal{D}$ and $\gamma' \in \mathcal{D}'$ such that $\tilde{f} = \gamma' \circ \tilde{f}' \circ \gamma$ and $\tilde{g} = \gamma' \circ \tilde{g}' \circ \gamma$. The equivalence class by conjugacy is called a lifting class.

$$[\tilde{f}, \tilde{g}] = \{(\gamma' \circ \tilde{f} \circ \gamma, \gamma' \circ \tilde{g} \circ \gamma) \mid \gamma \in \mathcal{D}, \gamma' \in \mathcal{D}'\}.$$

Theorem 2.5. (i) $\Gamma(f, g) = \bigcup_{(\tilde{f}, \tilde{g})} p\,\Gamma(\tilde{f}, \tilde{g})$

(ii) $p\,\Gamma(\tilde{f}, \tilde{g}) = p\,\Gamma(\tilde{f}', \tilde{g}')$ if $[\tilde{f}, \tilde{g}] = [\tilde{f}', \tilde{g}']$

(iii) $p\,\Gamma(\tilde{f}, \tilde{g}) \cap p\,\Gamma(\tilde{f}', \tilde{g}') = \phi$ if $[\tilde{f}, \tilde{g}] \neq [\tilde{f}', \tilde{g}']$.

Proof. (i) $x \in \Gamma(f, g)$, then $f(x) = g(x)$. Pick $\tilde{x} \in p^{-1}(x)$ and $\tilde{y} \in q^{-1}(f(x)) = q^{-1}(g(x))$, then there are unique liftings $\tilde{f}$, $\tilde{g}$ of f, g respectively, such that $\tilde{f}(\tilde{x}) = \tilde{g}(\tilde{x}) = \tilde{y}$ so that $\tilde{x} \in \Gamma(\tilde{f}, \tilde{g})$.

Conversely, if for some $\tilde{f}$ and $\tilde{g}$, $\tilde{f}(\tilde{x}) = \tilde{g}(\tilde{x})$ and $p(\tilde{x}) = x$, then we have $f(x) = f \circ p(\tilde{x}) = q \circ \tilde{f}(\tilde{x}) = g(x)$ so that $x \in \Gamma(f, g)$.

(ii) If $\tilde{f} = \gamma' \circ \tilde{f}' \circ \gamma$ and $\tilde{g} = \gamma' \circ \tilde{g}' \circ \gamma$, then by Lemma 2.3, $\Gamma(\tilde{f}, \tilde{g}) = \gamma^{-1}\Gamma(\tilde{f}', \tilde{g}')$ so that $p\Gamma(\tilde{f}, \tilde{g}) = p\Gamma(\tilde{f}', \tilde{g}')$.

(iii) If $x \in p\Gamma(\tilde{f}, \tilde{g}) \cap p\Gamma(\tilde{f}', \tilde{g}')$, then $\tilde{f}(\tilde{x}_0) = \tilde{g}(\tilde{x}_0)$ and $\tilde{f}'(\tilde{x}_1) = \tilde{g}'(x_1)$ for some $\tilde{x}_0$, $\tilde{x}_1 \in p^{-1}(x)$. By Lemma 2.2(i), there is $\gamma \in \mathcal{D}$ such that $\gamma(\tilde{x}_0) = \tilde{x}_1$ so that $\tilde{f}'\gamma(\tilde{x}_0) = \tilde{g}'\gamma(\tilde{x}_0)$.

Hence we have $[\tilde{f}, \tilde{g}] = [\tilde{f}', \tilde{g}']$ by Lemma 2.4.

Definition 2.6. The subset $p\Gamma(\tilde{f}, \tilde{g})$ of $p\Gamma(f, g)$ is called the coincidence class of f and g determined by the lifting class $[\tilde{f}, \tilde{g}]$. The set of coincidence classes is denoted by $\tilde{\Gamma}(f, g)$. The coincidence set $\Gamma(f, g)$ splits into a disjoint union of coincidence classes.

Example. If $\Gamma(f, g) = X$, then $p\Gamma(\tilde{f}, \tilde{g}) = \begin{cases} X, & \text{if } \Gamma(\tilde{f}, \tilde{g}) = \tilde{X} \\ \phi, & \text{otherwise.} \end{cases}$

Theorem 2.7. x_0, $x_1 \in \Gamma(f, g)$ belong to the same coincidence class iff there is a path c from x_0 to x_1 such that $f \circ c \simeq g \circ c$ (homotopic relative to end points).

Proof. Suppose x_0, $x_1 \in p\Gamma(\tilde{f}, \tilde{g})$. Then $\tilde{f}(\tilde{x}_0) = \tilde{g}(\tilde{x}_0)$ and $\tilde{f}(\tilde{x}_1) = \tilde{g}(\tilde{x}_1)$ for some $\tilde{x}_0 \in p^{-1}(x_0)$, $\tilde{x}_1 \in p^{-1}(x_1)$. Take a path $\tilde{c}$ in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{x}_1$. Since $\tilde{Y}$ is simply connected, $\tilde{f} \circ \tilde{c} \simeq \tilde{g} \circ \tilde{c}$.

Thus $f\circ c = q\circ\tilde{f}\circ\tilde{c} \simeq q\circ\tilde{g}\circ\tilde{c} = g\circ p\circ\tilde{c} = g\circ c$ where $p\circ\tilde{c} = c$.

Conversely, suppose $x_0 \in p\Gamma(\tilde{f}, \tilde{g})$, $\tilde{x}_0 \in p^{-1}(x_0)$ and $\tilde{f}(\tilde{x}_0) = \tilde{g}(\tilde{x}_0)$. We want to prove that $x_1 \in p\Gamma(\tilde{f}, \tilde{g})$. By the covering path property, there is a unique covering path $\tilde{c}$ of c such that $\tilde{c}(0) = \tilde{x}_0$.

Then $\tilde{f}\circ\tilde{c}$ and $\tilde{g}\circ\tilde{c}$ are covering paths of $f\circ c$ and $g\circ c$ from $\tilde{f}(\tilde{x}_0)$ and $\tilde{g}(\tilde{x}_0)$, respectively. Thus $\tilde{f}\circ\tilde{c} \simeq \tilde{g}\circ\tilde{c}$ and $\tilde{f}\circ\tilde{c}(1) = \tilde{g}\circ\tilde{c}(1)$ so that $\tilde{f}(\tilde{x}_1) = \tilde{g}(\tilde{x}_1)$ and $p(\tilde{x}_1) = x_1$, where $\tilde{c}(1) = \tilde{x}_1$.

Definition 2.8. A homotopy $\tilde{F} = \{\tilde{f}_t\} : \tilde{X} \to \tilde{Y}$ is called a lifting of the homotopy $F = \{f_t\} : X \to Y$ if $\tilde{f}_t$ is a lifting of f_t for each $t \in I$.

Let $F : f_0 \simeq f_1$ and $G : g_0 \simeq g_1$ be homotopies from X to Y. Given a lifting $\tilde{f}_0$ of f_0, there is a unique lifting $\tilde{F}$ of F such that $\tilde{F}(0) = \tilde{f}_0$, hence they determine a lifting $\tilde{f}_1$ of f_1.
Thus F gives rise to a one-to-one correspondence from lifting of f_0 to lifting of f_1.

$$\tilde{f}_0 \xrightarrow{\ F\ } \tilde{f}_1 \xrightarrow{\ F^{-1}\ } \tilde{f}_0$$

Similarly, G gives rise to a one-to-one correspondence from liftings of g_0 to liftings of g_1. Thus F and G give rise to a one-to-one correspondence from pairs of liftings of f_0 and g_0 to pairs of liftings of f_1 and g_1.

$$(\tilde{f}_0, \tilde{f}_0) \xrightarrow{\ (F,\ G)\ } (\tilde{f}_1, \tilde{g}_1) \xrightarrow{\ (F^{-1},\ G^{-1})\ } (\tilde{f}_0, \tilde{g}_0)$$

This correspondence preserves the conjugacy relation :

$$\{\tilde{f}_t\} : \ \tilde{f}_0 \simeq \tilde{f}_1 \text{ implies } \{\gamma'\circ\tilde{f}_t\circ\gamma\} : \ \gamma'\circ\tilde{f}_0\circ\gamma \simeq \gamma'\circ\tilde{f}_1\circ\gamma$$
$$\{\tilde{g}_t\} : \ \tilde{g}_0 \simeq \tilde{g}_1 \text{ implies } \{\gamma'\circ\tilde{g}_t\circ\gamma\} : \ \gamma'\circ\tilde{g}_0\circ\gamma \simeq \gamma'\circ\tilde{g}_1\circ\gamma.$$

Thus we have

Theorem 2.9. If $f_0 \simeq f_1$ and $g_0 \simeq g_1$, then there is a one-to-one correspondence between the lifting classes of f_0 and g_0 (and the coincidence classes of f_0 and g_0) and the lifting classes of f_1 and g_1 (and the coincidence classes of f_1 and g_1). Hence the number of coincidence classes is homotopy invariant.

Definition 2.10. Let $F : f_0 \simeq f_1$ and $G : g_0 \simeq g_1$ be homotopies from X to Y and $\tilde{f}_i$, $\tilde{g}_i$ be lifting of f_i, g_i, i=0, 1.

We say that the coincidence class $p\Gamma(\tilde{f}_0, \tilde{g}_0)$ corresponds to $p\Gamma(\tilde{f}_1, \tilde{g}_1)$ via F and G if F and

G have liftings $\widetilde{F} : \widetilde{f}_0 \simeq \widetilde{f}_1$ and $\widetilde{G} : \widetilde{g}_0 \simeq \widetilde{g}_1$, respectively.

Definition 2.11. Suppose F a homotopy from X to Y and c a path in X. Then $\langle F, c\rangle$ is the path in Y defined by $\langle F, c\rangle(t) = F(t)(c(t))$, $t \in I$.

Theorem 2.12. Suppose that $F : f_0 \simeq f_1$ and $G : g_0 \simeq g_1$ are homotopies from X to Y and $\widetilde{f}_i$, $\widetilde{g}_i$ be liftings of f_i, g_i (i=0, 1) and that $x_0 \in p\Gamma(\widetilde{f}_0, \widetilde{g}_0)$ and $x_1 \in p\Gamma(\widetilde{f}_1, \widetilde{g}_1)$. Then, $p\Gamma(f_0, g_0)$ corresponds to $p\Gamma(f_1, g_1)$ via F, G iff there is a path c from x_0 to x_1 such that $\langle F, c\rangle \simeq \langle G, c\rangle$ (homotopic relative to end points).

Proof. Suppose $p\Gamma(\widetilde{f}_0, \widetilde{g}_0)$ corresponds to $p\Gamma(\widetilde{f}_1, \widetilde{g}_1)$ via F, G. Then by definition, F and G have lifting $\widetilde{F} : \widetilde{f}_0 \simeq \widetilde{f}_1$ and $\widetilde{G} : \widetilde{g}_0 \simeq \widetilde{g}_1$, respectively. Since $x_0 \in p\Gamma(\widetilde{f}_0, \widetilde{g}_0)$ and $x_1 \in p\Gamma(\widetilde{f}_1, \widetilde{g}_1)$, $\widetilde{f}_0(\widetilde{x}_0) = \widetilde{g}_0(\widetilde{x}_0)$ and $\widetilde{f}_1(\widetilde{x}_1) = \widetilde{g}_1(\widetilde{x}_1)$ for some $\widetilde{x}_0 \in p^{-1}(x_0)$ and $\widetilde{x}_1 \in p^{-1}(x_1)$. Take a path $\widetilde{c}$ in $\widetilde{X}$ from $\widetilde{x}_0$ to $\widetilde{x}_1$, then $p \circ \widetilde{c}$ is a path in X from x_0 to x_1 and $\langle \widetilde{F}, \widetilde{c}\rangle$ and $\langle \widetilde{G}, \widetilde{c}\rangle$ are paths in Y from $\widetilde{f}_0(\widetilde{x}_0) = \widetilde{g}_0(\widetilde{x}_0)$ to $\widetilde{f}_1(\widetilde{x}_1) = \widetilde{g}_1(\widetilde{x}_1)$. Since $\widetilde{Y}$ is simply connected, $\langle \widetilde{F}, \widetilde{c}\rangle$ and $\langle \widetilde{G}, \widetilde{c}\rangle$ are homotopic. Thus $q\langle \widetilde{F}, \widetilde{c}\rangle$ and $q\langle \widetilde{G}, \widetilde{c}\rangle$ are homotopic and so are $\langle F, c\rangle$ and $\langle G, c\rangle$, where $c = p \circ \widetilde{c}$.

Conversely, let $\widetilde{F} : \widetilde{f}_0 \simeq \widetilde{f}$ and $\widetilde{G} : \widetilde{g}_0 \simeq \widetilde{g}$ be the liftings of F and G, respectively. Then $p\Gamma(\widetilde{f}_0, \widetilde{g}_0)$ corresponds to $p\Gamma(\widetilde{f}, \widetilde{g})$ via F, G. We want to prove that $p\Gamma(\widetilde{f}_1, \widetilde{g}_1) \cap p\Gamma(\widetilde{f}, \widetilde{g}) \neq \phi$.

Let $\widetilde{c}$ be the unique lifting path of c^{-1} such that $\widetilde{c}(0) = \widetilde{x}_1$, where $\widetilde{x}_1 \in \Gamma(\widetilde{f}_1, \widetilde{g}_1)$ and $p(\widetilde{x}_1) = x_1$. Then $\langle \widetilde{F}, \widetilde{c}^{-1}\rangle$ and $\langle \widetilde{G}, \widetilde{c}^{-1}\rangle$ are unique lifting paths of $\langle F, c\rangle$ and $\langle G, c\rangle$ from $\widetilde{f}_0 \circ \widetilde{c}(1)$ and $\widetilde{g}_0 \circ \widetilde{c}(1)$, respectively. Since $\langle F, c\rangle \simeq \langle G, c\rangle$, we have $\langle \widetilde{F}, \widetilde{c}^{-1}\rangle \simeq \langle \widetilde{G}, \widetilde{c}^{-1}\rangle$ and $\langle \widetilde{F}, \widetilde{c}^{-1}\rangle(1) = \langle \widetilde{G}, \widetilde{c}^{-1}\rangle(1)$ so that $\widetilde{f}(\widetilde{x}_1) = \widetilde{g}(\widetilde{x}_1)$. Thus $p\Gamma(\widetilde{f}_1, \widetilde{g}_1) \cap p\Gamma(\widetilde{f}, \widetilde{g}) \neq \phi$.

Example. A non-empty coincidence class may disappear under homotopics. Consider maps $S^1 \to S^1$. The universal covering is $p : R \to S^1$, given by $\theta \to e^{i\theta}$.

Let $F = \{f_t : Z \to Ze^{it\epsilon}\}$ and $G = \{g_t : z \to z\}$ ($\epsilon > 0$). Taking the liftings $\widetilde{F} = \{\widetilde{f}_t : \theta \to \theta + t\epsilon\}$ and $\widetilde{G} = \{\widetilde{g}_t : \theta \to \theta\}$ of F and G, respectively, then $p\Gamma(\widetilde{f}_0, \widetilde{g}_0) = S^1$ but $p\Gamma(\widetilde{f}_1, \widetilde{g}_1) = \phi$.

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