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Some Properties of Analytical Indices on K(B(M), S(M))

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In this note, we shall assume that M is a closed orented Riemannian manifold of dimension n without any statements. Let T(M) be the tangent bundle over M. For

$$B(M) = \{v \in T(M) \mid \|v\| \leq 1\},\ S(M) = \{v \in T(M) \mid \|v\| = 1\}$$

let K(B(M), S(M)) be the relative K-groups on (B(M), S(M)).

The purpose of this note is to prove some properties of elliptic differential operators (Proposition 4, Lemma 5) and to prove that there exists a homomorphism

$$i_a : K(B(M), S(M)) \to \mathbf{Z}$$

(Theorem 7) which is induced from the Atiyah-Singer analytic indices ([2]).

Let ξ and η be two complex vector bundles over M. We put

$$\varepsilon(M, \xi) = \{f : M \to \xi \mid f \text{ is a } C^\infty\text{-cross section}\}.$$

For the Sobolev norm $\|\ \|_{s,\xi}$ let $W^s(M, \xi)$ be the completion of $\varepsilon(M, \xi)$ with respect to the norm $\|\ \|_{s,\xi}$ ([1]). Then we have a sequence of inclusions of the Hilbert spaces

$$\cdots \supset W^s \supset W^{s+1} \supset \cdots \supset W^{s+j} \supset \cdots$$

where $W^s = W^s(M, \xi)$ ([1]). If $t<s$ then the natural inclusion $j : W^s(M, \xi) \to W^t(M, \xi)$ is a completely continuous linear map. We define $OP_k(\xi, \eta)$ the set of all C-linear map $T : \varepsilon(M,\xi) \to \varepsilon(M,\eta)$ such that there exists the extension $\bar{T} : W^s(M,\xi) \to W^{s-k}(M,\eta)$ of T which is called an *operator of order k*, where C is the field of complexes.

We shall define a subspace $Int_k(\xi,\eta)$ of $OP_k(\xi, \eta)$ with the following properties ([4]).

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(i) $Op_{k-1}(\xi,\eta) \lneq Int_k(\xi,\eta)$ which is precisely the kernel of the linear symbol map

$$\sigma_k : Int_k(\xi,\eta) \to Smbl_k(\xi,\eta).$$

(ii) Let $Diff_k(\xi,\eta)$ be the set of differential operators from $\varepsilon(M,\xi)$ to $\varepsilon(M,\eta)$ with rank k. Then $Diff_k(\xi,\eta)$ $Int_k(\xi,\eta)$ and σ_k is an extension of the symbol maps $Diff_k(\xi,\eta) \to Smbl_k(\xi,\eta)$.

(iii) $T \in Int_k(\xi,\eta)$ and $S \in Int_l(\eta,\zeta)$ (ζ is a complex vector bundle over M) $\Rightarrow$ $ST \in Int_{k+l}(\xi,\eta)$ and $\sigma_{k+l}(ST) = \sigma_l(S)\sigma_k(T)$.

(iv) For each $T \in Int_k(\xi,\eta)$ the transpose $T^t \in Int_k(\eta,\xi)$ and $\sigma_k(T^t) = (-1)^k\sigma_k(T)^*$.

(v) There is the continuous right inverse

$$X_k : Smbl_k(\xi,\eta) \to Int_k(\xi,\eta).$$

We define the projection by

$$\pi : T'(M) = (T(M)-M) \to M$$

and put such that $\tilde{\xi} = \pi^*\xi \mid S(M)$. We also use the following notations :

$L(\xi,\eta)$ = the complex vector bundle over S(M) whose fiber at $(v,x) \in S(M)$ is the space of all linear maps of ξ_x into η_x.

$I(\tilde{\xi},\tilde{\eta})$ = the open subspace of $L(\tilde{\xi},\tilde{\eta})$ whose fiber at $(v,x) \in S(M)$ is all linear isomorphisms of ξ_x into η_x.

$H(\tilde{\xi},\tilde{\eta})$ = the complex vector bundle of all C^∞-cross sections of $L(\tilde{\xi},\tilde{\eta})$, i.e.,

$$Hom(\tilde{\xi},\tilde{\eta}) = C^\infty(L(\tilde{\xi},\tilde{\eta}))$$

$Iso(\tilde{\xi},\tilde{\eta}) = C^\infty(I(\tilde{\xi},\tilde{\eta}))$.

Definition 1. For $h_t \in Iso(\tilde{\xi},\tilde{\eta})$ ($t \in [0, 1]$ if there is a C^∞-function $h : S(M) \times [0, 1] \to Iso(\tilde{\xi},\tilde{\eta})$ defined by

$$h((v,x),t) = h_t(v,x)$$

then $\{h_t\}$ is called a *regular homotopy of h_0 with h_1*. Moreover, h_0 and h_1 are C^∞-*regularly homotopic*.

Property 2. (i) $\sigma \in Smbl_k(\xi,\eta)$ is elliptic if and only if $\tilde{\sigma} = \sigma|S(M) \in Iso(\tilde{\xi},\tilde{\eta})$ which is dense in $C^0(I(\tilde{\xi},\tilde{\eta})) = \{f : M \to I(\tilde{\xi},\tilde{\eta}) \mid f \text{ is continuous}\}$([4]).

(ii) $C^\infty(L(\tilde{\xi},\tilde{\eta})) = Hom(\tilde{\xi},\tilde{\eta})$ is dense in $C^0(L(\tilde{\xi},\tilde{\eta}))$ with is a Banach space consisting of all continuous cross sections of $L(\tilde{\xi},\tilde{\eta})$ with the compact open topology. Moreover, $C^0(I(\tilde{\xi},\tilde{\eta}))$ is open in the locally arcwise connected Banach space $C^0(L(\tilde{\xi},\tilde{\eta}))$ ([4]).

Definition 3. Let $\triangle(\tilde{\xi},\tilde{\eta})$ be the set of arc components of $C^0(I(\tilde{\xi},\tilde{\eta})) \subset C^0(L(\tilde{\xi},\tilde{\eta}))$. For each $\sigma \in C^0(I(\tilde{\xi},\tilde{\eta}))$ let $[\sigma]$ be the arc component in $\triangle(\tilde{\xi},\tilde{\eta})$ to which σ belongs. For σ, $\sigma' \in C^0(I(\tilde{\xi},\tilde{\eta}))$ if $[\sigma] = [\sigma']$ then we say that σ and σ' are *homotopic*.

Proposition 4. The map

$$\mathrm{Iso}(\tilde{\xi},\tilde{\eta}) \rightarrow \triangle(\xi,\eta) \quad (\sigma' \rightarrow [\sigma])$$

is surjective. Moreover, if $[\sigma] = [\sigma']$ $(\sigma,\sigma' \in C^0(I(\tilde{\xi},\tilde{\eta})))$ then they are C^∞−regularly homotopic.

Proof. By (ii) of Property 2 for each component δ of $C^0(L(\tilde{\xi},\tilde{\eta}))$ $\delta \cap C^0(I(\tilde{\xi},\tilde{\eta}))$ is open in $C^0(I(\tilde{\xi},\tilde{\eta}))$. Moreover, since $\mathrm{Iso}(\tilde{\xi},\tilde{\eta})$ is dense in $C^0(I(\tilde{\xi},\tilde{\eta}))$ by (i) of property 2 we have an element

$$\sigma \in \mathrm{Iso}(\tilde{\xi},\tilde{\eta}) \cap \delta \circ\in\circ [\sigma] = \delta.$$

Thus, $\sigma \rightarrow [\sigma] = \delta$ is surjective.

Next, we have to note that $C^\infty(I(\tilde{\xi}\times[0,1],\tilde{\eta}\times[0,1]))$ is dense in $C^0(I(\tilde{\xi}\times[0,1], \tilde{\eta}\times[0,1]))$ by the same reason that $C^\infty(I(\tilde{\xi},\tilde{\eta}))$ is dense in $C^0(I(\tilde{\xi},\tilde{\eta}))$. Since $[\sigma] = [\sigma']$ there exists a C^∞−map

$$h_t : S(M) \rightarrow \mathrm{Iso}(\tilde{\xi},\tilde{\eta})$$

such that $h_0 = \sigma$ and $h_1 = \sigma'$. Hence

$$h : S(M) \times [0,1] \rightarrow \mathrm{Iso}(\tilde{\xi},\tilde{\eta})$$

is defined by $h(v,x),t) = h_t(v,x)$. By Definition 1 σ and σ' are C^∞−regularly homotopic. / / /

We shall put such that

$$E_k(\xi,\eta) = \{T \in \mathrm{Int}_k(\xi,\eta) \mid \sigma_k(T) \text{ is elliptic}\}$$

and

$$E(\xi,\eta) = \cup_{k\in\mathbf{Z}} E_k(\xi,\eta) \qquad (\mathbf{Z}\text{=the integers})$$

Thus $E_k(\xi,\eta) \cap \mathrm{Diff}_k(\xi,\eta)$ is just the set of K^{th} order elliptic differential operators from ξ to η. We define for each elliptic operator $T \in E_k(\xi,\eta)$ the *index* $\mathrm{Ind}(T)$ *of* T by $\mathrm{Ind}(T) = \dim \ker T - \dim \mathrm{Coker}\, T < \infty$ ([1], [4]).

For each symbol map $\sigma \in \mathrm{Smbl}_k(\xi,\eta)$ we put $\tilde{\sigma} = \sigma|S(M)$. In this case, σ is elliptic if and only if $\tilde{\sigma} \in \mathrm{Iso}(\tilde{\xi},\tilde{\eta})$ ([4]). We have the restricted symbol map

$$\Sigma : E(\xi,\eta) \to \mathrm{Iso}(\tilde{\xi},\tilde{\eta})$$

which is defined by $\Sigma(T) = \tilde{\sigma}_k(T)$.

Lemma 5. There is a surjective map

$$\delta : E(\xi,\eta) \to \triangle(\xi,\eta),$$

and if $\delta(S)=\delta(T)$ for $S,T \in E(\xi,\eta)$ then $\mathrm{Ind}(S)=\mathrm{ind}(T)$.

Proof. By Proposition 4 we can find $\sigma_0 \in \mathrm{Iso}(\tilde{\xi},\tilde{\eta})$ for each $\delta_0 \in \triangle(\xi,\eta)$ such that $[\sigma_0] = \delta_0$. Since

$$\mathrm{Smbl}_k(\xi,\eta) \to \mathrm{Hom}(\tilde{\xi},\tilde{\eta}) \quad (\sigma \mapsto \tilde{\sigma})$$

is a bijective ([4]) there is $\sigma \in \mathrm{Smbl}_k(\xi,\eta)$ which is elliptic such that $\tilde{\sigma}=\sigma_0$. Since the sequence

$$0 \to OP_{k-1}(\xi,\eta) \to \mathrm{Int}_k(\xi,\eta) \xrightarrow{\sigma_k} \mathrm{Smbl}_k(\xi,\eta) \to 0$$

is exact ([4]) there exists $T \in E_k(\xi,\eta)$ such that $\sigma_k(T)=\sigma$. Then

$$\delta(T)=[\Sigma(T)]=[\tilde{\sigma}_k(T)]=[\tilde{\sigma}]=[\sigma_0]=\delta_0$$

proving the first statement.

We assume that $\delta(T)=[\Sigma(T)]=\delta(S)=[\Sigma(S)]$. Then by Proposition 4 $\Sigma(S)$ and $\Sigma(T)$ are C^∞-regularly homotopic, and thus $\mathrm{Ind}(S)=\mathrm{Ind}(T)$ ([4]). / / /

Property 6. For every elliptic differential operator $D : C^\infty(\xi) \to C^\infty(\eta)$ there is an elliptic differential operator D' with order zero such that $\sigma(D)=\sigma(D')$ ([4]) where $\sigma : \mathrm{Diff}_k(\xi,\eta) \to \mathrm{Smbl}_k(\xi,\eta)$ is the symbol map.

For each element $\gamma_0 \in K(B(M), S(M))$ there exist complex vector bundles ξ and η over M and an isomorphism

$$\sigma_0 : \pi^*\xi|S(M) \xrightarrow{\cong} \pi^*\eta|\ S(M)$$

such that $\gamma_0=d(\pi^*\xi,\ \pi^*\eta,\sigma_0)$ which is a difference element ([1],[3]). By Lemma 5 we have an elliptic differential operator $D : C^\infty(\xi) \to C^\infty(\eta)$ such that $\sigma(D)$ is homotopic to σ_0(i.e., $[\sigma(D)]$

$=[\sigma_0]$).

Hence we have $\gamma_0=d(\pi^*\xi,\ \pi^*\eta,\ \sigma(D))$. That is, γ_0 depends only on D, ξ and η, and thus we put.

$$\gamma_0 = \gamma(D) = d^*(\pi^*\xi,\pi^*\eta,\sigma(D)).$$

Theorem 7. There is a homomorphism

$$i_a : K(B(M),\ S(M)) \to Z$$

such that each $\gamma(D) = d(\pi^*\xi,\pi^*\eta,\ \sigma(D)) \in K(B(M),S(M))$ $i_a(\gamma(D))=i_a(D)$, where $D : C^\infty(\xi) \to C^\infty(\eta)$ is an elliptic differential operator.

Proof. By Property 6, it suffices to consider only elliptic differential operator D with order zoro.

(i) We shall prove that $\gamma(D)=0$ implies that $i_a(\gamma(D))=i_a(D)=0$. At firtst. we assume that $\sigma(D)$ extends to an automorphism of $\pi^*\xi\cong\pi^*\eta$ over all of B(M). Then $\gamma(D)=0$ ([3],[4]). We want to show that $i_a(D)=0$. Since π:B(M) $\to$ M is a homotopy equivalence, for some isomorphism $\varphi:\xi\to\eta$ $\sigma(D)$ is homotopic to $\pi^*\varphi$. Sine $\varphi_*:C^\infty(\xi) \to C^\infty(\eta)$ is an elliptic differential operator of order zoro and $\sigma(\varphi_*)=\pi^*\varphi|S(M)$ we have $i_a(D)=i_a(\varphi_*)=0$(Note that φ_* is an isomorphism).

Next, we shall prove that if $\gamma(D)=0$ for some elliptic differential operator D of order zero then $i_a(D)=0$. Since there is a bundle ξ over M such that

$$\sigma(D) \oplus 1_{\tilde{\zeta}} : \tilde{\xi} \oplus \tilde{\zeta} \cong \tilde{\eta} \oplus \tilde{\zeta}$$

extends to an isomorphism over all of B(M) ([3],][4]). Since $\sigma(D)\oplus 1_\zeta = \sigma(D\oplus 1_\zeta)$, by the above reason we have $i_a(D\oplus 1_\zeta)=0$. But

$$0=i_a(D\oplus 1_\zeta)=i_a(D)+i_a(1_\zeta)=i_a(D)$$

since $i_a(1_s)=0$.

(ii) We assume that for elliptic operators D_1 and D_2 $\gamma(D_1)=\gamma(D_2)$. We shall prove that $i_a(D_1)=i_a(D_2)$. Let D_3 be elliptic of order zero such that $\gamma(D_3)=-\gamma(D_1)$ ([3]). Then by the lineality of the symbol map σ we have

$$\sigma(D_1\oplus D_3)=\sigma(D_1)+\sigma(D_3)$$

([3]). Since

$$\gamma(D_1\oplus D_3)=\gamma(D_1)+\gamma(D_3)=0$$

and thus by (i)

$$0=i_a(D_1\oplus D_3)=i_a(D_1)+i_a(D_3).$$

Similarly, from $\gamma(D_2\oplus D_3)=\gamma(D_2)+\gamma(D_3)=0$ we have

$$0=i_a(D_2\oplus D_3)=i_a(D_2)+i_a(D_3).$$

Hence

$$0=i_a(D_1)+i_a(D_3)=i_a(D_2)+i_a(D_3)$$

and thus $i_a(D_1)=i_a(D_2)$. Therefore

$$i_a(D)=i_a(\gamma(D))$$

is well-defined on K(B(M), S(M)). Moreover, by the additivity of the difference construction i_a is a homomorphism. / / /

References

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