Comm. Korean Math. Soc. 7 (1992), No. 2, pp. 173–188

A GENERAL THEORY OF SOME NEGATIVE DEPENDENCE NOTIONS

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1. Introduction

The concept of association was introduced into the statistical literature by Esary, Proschan, and Walkup(1967). Since then a great many papers have been written on the subject and its extensions, and numerous multivariate inequalities have been obtained.

It is well known(see Esary et al. [2]) that X is associated if and only if

$$P(\underset{\sim}{X} \in A \cap B) \geq P(\underset{\sim}{X} \in A)\, P(\underset{\sim}{X} \in B) \tag{1.1}$$

whenever A and B are open upper sets(U is an upper set if $\underset{\sim}{x} \in$ U and $\underset{\sim}{y} \geq \underset{\sim}{x}$ imply $\underset{\sim}{y} \in$ U). In addition, Shaked(1982) find that a possible way of weakening the condition of association is to require that (1.1) holds for all $\mathcal{A}$ and $\mathcal{B}$ which belong to a subcollection of the collection of all open upper sets, that is, let A and B be two collections of sets in R^n then the random vector $\underset{\sim}{X}$ is positively dependent relative to A and B(denoted by PD($\mathcal{A},\mathcal{B}$)) if

$$P(\underset{\sim}{X} \in A \cap B) \geq P(\underset{\sim}{X} \in A) P(\underset{\sim}{X} \in B) \tag{1.2}$$

whenever A $\in \mathcal{A}$ and B $\in \mathcal{B}$.

Joag-Dev and Proschan(1983) introduced the notions of negative association, derived basic theoretical properties, and developed applications in multivariate statistical analysis. It is well known that the $\underset{\sim}{X}$'s are negatively associated then they are negatively upper orthant dependent (NUOD) and negatively lower orthant dependent (NLOD)(see Joag-Dev

Received March 11, 1991. Revised September 17, 1991.
This work is partially supported by KOSEF Grant.

and Proschan [3]). Various results in probability and statistics have been derived under the assumption that some underlying random variables are negatively associated. In some cases(see, e.g Remark 3.3) , a careful inspection of the proofs of these results indicates that the results are valid even if one weakens the assumption of negative association, however, the validity of the proofs may be violated if instead of the assumption of negative association one merely assumes NUOD or NLOD. Thus, various notions of negative dependence, which are between negative association and the orthant dependence notions may be useful.

The purpose of this paper is to derive various notions of negative dependence which are weaker than negative association but stronger than negative orthant dependence by using arguments similar to those of Shaked(1982) and to investigate their interrelationships motivated by (1.1) and (1.2).

The general propositions and some definitions are given in Section 2. In Section 3 the specialized negative dependence are developed and their interrelationships are derived from the concepts of the general propositions . In Section 4 we introduce some concepts of functional negative dependence(FND) and show that for j=1,...,5 the notion of ND($\mathcal{A}_j$) essentially implies the notion of FND($\mathcal{F}_j$).

2. Definitions and General Propositions

DEFINITION 2.1. (Joag - Dev and Proschan, 1983). A random vector $\underline{X} = (X_1, \ldots, X_n)$ is said to be negatively associated (NA) if for every pair of partitions $\underline{X}_1 = (X_{\pi(1)}, \ldots, X_{\pi(k)})$ and $\underline{X}_2 = (X_{\pi(k+1)}, \ldots, X_{\pi(n)})$ of $\underline{X}$, and for every pair of increasing functions $f: R^k \longrightarrow R$, $g: R^{n-k} \longrightarrow R$

$$\text{(2.1)} \qquad \mathrm{Cov}(f(\underline{X}_1), g(\underline{X}_2)) \leq 0$$

whenever π is any permutation of $\{1, \ldots, n\}$, $1 \leq k \leq n-1$

DEFINITION 2.2. (Joag - Dev and Proschan, 1983). A random vector $\underline{X} = (X_1, \ldots, X_n)$ is said to be negatively upper orthant dependent (NUOD) if for every real vector $\underline{x} = (x_1, \ldots, x_n)$,

$$\text{(2.2)} \qquad \underline{P}(\underline{X} > \underline{x}) \leq \prod_{i=1}^{n} \underline{P}(X_i > x_i)$$

and it is negatively lower orthant dependent(NLOD) if for every real vector $\underline{x} = (x_1, \ldots, x_n)$,

$$P(\underline{X} \leq \underline{x}) \leq \prod_{i=1}^{n} P(X_i \leq x_i) \tag{2.3}$$

Moreover, if $\underline{X}$ is NUOD and NLOD then $\underline{X}$ is negatively orthant dependent(NOD). When n = 2 then $\underline{X} = (X_1, X_2)$ is NUOD if and only if $\underline{X}$ is NLOD(see Lehmann [5]); we say then that $\underline{X}$ is negatively quadrant dependent(NQD).

DEFINITION 2.3. (Joag - Dev, 1983). A random vector $\underline{X}$ is said to be linearly negatively quadrant dependent(LNQD) if for every pair of nonnegative vectors $\underline{r} = (r_1, \ldots, r_k)$, $\underline{s} = (s_1, \ldots, s_{n-k})$ and every pair of partitions $\underline{X}_1 = (X_{\pi(1)}, \ldots, X_{\pi(k)})$, $\underline{X}_2 = (X_{\pi(k+1)}, \ldots, X_{\pi(n)})$ of $\underline{X}$, $\underline{r} \cdot \underline{X}_1$, $\underline{s} \cdot \underline{X}_2$ is NQD whenever π denotes any permutation of { 1,. . . ,n } and k = 1,. . . ,n-1.

The concept of LNQD is similar to one of negative dependence which will be discussed in later section (see Theorem 3.4).

Motivated by (1.1) we have the following equivalent notion of negative association.

PROPOSITION 2.4. *A random vector $\underline{X} = (X_1, \ldots, X_n)$ is negatively associated if and only if for every pair of partitions $\underline{X}_1 = (X_{\pi(1)}, \ldots, X_{\pi(k)})$, $\underline{X}_2 = (X_{\pi(k+1)}, \ldots, X_{\pi(n)})$ of $\underline{X}$*

$$P(\underline{X}_1 \in A, \underline{X}_2 \in B) \leq P(\underline{X}_1 \in A) P(\underline{X}_2 \in B) \tag{2.4}$$

whenever A and B are open upper sets, $1 \leq k \leq n-1$, and π is any permutation of { 1,. . . ,n }.

Proof. We only show the converse: let π be any permutation of { 1,. . . ,n }, $\underline{X}_1 = (X_{\pi(1)}, \ldots, X_{\pi(k)})$, $\underline{X}_2 = (X_{\pi(k+1)}, \ldots, X_{\pi(n)})$ be arbitrary partitions of $\underline{X}$, and f, g be arbitrary increasing functions of $\underline{X}_1$, $\underline{X}_2$, respectively. Then for every real s and t, A = { $f(\underline{X}_1) > s$ } and B = { $g(\underline{X}_2) > t$ } are open upper sets. Thus

$$\begin{aligned} P(f(\underline{X}_1) > s, g(\underline{X}_2) > t) &= P(\underline{X}_1 \in A, \underline{X}_2 \in B) \\ &\leq P(\underline{X}_1 \in A) P(\underline{X}_2 \in B) \\ &= P(f(\underline{X}_1) > s) P(g(\underline{X}_2) > t). \end{aligned} \tag{2.5}$$

Define

$$X_f(s) = \begin{cases} 1 & \text{if } f(\underset{\sim}{X}_1) > s \\ 0 & \text{otherwise,} \end{cases} \qquad X_g(t) = \begin{cases} 1 & \text{if } g(\underset{\sim}{X}_2) > t \\ 0 & \text{otherwise} \end{cases}.$$

Then

(2.6)

$$\begin{aligned} &\mathrm{Cov}(f(\underset{\sim}{X}_1), g(\underset{\sim}{X}_2)) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{Cov}(X_f(s), X_g(t))\, ds\, dt \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [E(X_f(s)X_g(t)) - EX_f(s)EX_g(t)]\, ds\, dt \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\underset{\sim}{P}(f(\underset{\sim}{X}_1) > s, g(\underset{\sim}{X}_2) > t) - \underset{\sim}{P}(f(\underset{\sim}{X}_1) > s)\underset{\sim}{P}(g(\underset{\sim}{X}_2) > t)]\, ds\, dt \end{aligned}$$

Thus (2.5) and (2.6) yield Cov($f(\underset{\sim}{X}_1)$, $g(\underset{\sim}{X}_2)) \le 0$. †

A possible way relaxing the condition of negative association is to require that (2.4) holds for all A and B which belong to subcollections of the collection of all upper sets in R^k and R^{n-k}, respectively. This will be the approach in this paper.

Let $\mathcal{A}^{(k)}$ be a collection of sets in R^k and $\mathcal{A}^{(n-k)}$ be a collection of sets in R^{n-k} (k = 1,...,n-1).

DEFINITION 2.5. A random vector $\underset{\sim}{X} = (X_1,\ldots,X_n)$ is negatively dependent relative to $\mathcal{A}^{(k)}$ and $\mathcal{A}^{(n-k)}$ (denoted by ND($\mathcal{A}^{(k)}$,$\mathcal{A}^{(n-k)}$)) if for any partitions $\underset{\sim}{X}_1 = (X_{\pi(1)},\ldots, X_{\pi(k)})$, $\underset{\sim}{X}_2 = (X_{\pi(k+1)},\ldots, X_{\pi(n)})$ of $\underset{\sim}{X}$

$$\underset{\sim}{P}(\underset{\sim}{X}_1 \in A, \underset{\sim}{X}_2 \in B) \le \underset{\sim}{P}(\underset{\sim}{X}_1 \in A)\underset{\sim}{P}(\underset{\sim}{X}_2 \in B) \tag{2.7}$$

whenever $A \in \mathcal{A}^{(k)}$ and $B \in \mathcal{A}^{(n-k)}$, π is any permutation of { 1,...,n } and a random vector $\underset{\sim}{X}$ is negatively dependent relative to $\mathcal{A}^{(n)}$ (denoted by ND($\mathcal{A}^{(n)}$)) if (2.7) holds for every k, where k = 1,...,n-1.

The following general propositions of negative dependence are motivated by those of positive dependence in Shaked(1982) but those are not duals of them, that is, in negative dependence case, we split the random vector into two subvector and consider the concepts of the negative dependence between them with the argument similar to that of Shaked(1982). These are easy to prove and will be used in later section.

PROPOSITION 2.6. *If* $\mathcal{A}^{(k)} \subset \tilde{\mathcal{A}}^{(k)}$ *and* $\mathcal{A}^{(n-k)} \subset \tilde{\mathcal{A}}^{(n-k)}$ *then* $ND(\tilde{\mathcal{A}}^{(k)}, \tilde{\mathcal{A}}^{(n-k)})$ *implies* $ND(\mathcal{A}^{(k)}, \mathcal{A}^{(n-k)})$ *and if* $\mathcal{A}^{(k)} \subset \tilde{\mathcal{A}}^{(k)}$ *,k=1,2,...,n-1, then* $ND(\tilde{\mathcal{A}}^{(n)})$ *implies* $ND(\mathcal{A}^{(n)})$.

Put $\bar{\mathcal{A}}^{(k)} = \{A\colon \bar{A} \in \mathcal{A}^{(k)}\}$ ($\bar{A}$ denotes the complement of A in R^k) and $-\mathcal{A}^{(k)} = \{A\colon -A \in \mathcal{A}^{(k)}\}$ ($-A$ denotes $\{\underline{X}\colon -\underline{X} \in A\}$).

PROPOSITION 2.7. *The random vector* $\underline{X}$ *is* $ND(\mathcal{A}^{(k)}, \mathcal{A}^{(n-k)})$ *if and only if* $\underline{X}$ *is* $ND(\bar{\mathcal{A}}^{(k)}, \bar{\mathcal{A}}^{(n-k)})$ *and the random vector* $\underline{X}$ *is* $ND(\mathcal{A}^{(n)})$ *if and only if* $\underline{X}$ *is* $ND(\bar{\mathcal{A}}^{(n)})$

Proof. Assume that $\underline{X}$ is $ND(\mathcal{A}^{(k)},\mathcal{A}^{(n-k)})$ for some k($k = 1,\ldots,n-1$). Let π be any permutation of $\{1,\ldots,n\}$ and $\underline{X}_1 = (X_{\pi(1)},\ldots,X_{\pi(k)})$, $\underline{X}_2 = (X_{\pi(k+1)},\ldots, X_{\pi(n)})$ be any pair of partitions of $\underline{X}$. Then for every $A \in \mathcal{A}^{(k)}$ and $B \in \mathcal{A}^{(n-k)}$

$$
\begin{aligned}
&\underline{P}(\underline{X}_1 \in A, \underline{X}_2 \in B)\\
&\quad \le \underline{P}(\underline{X}_1 \in A)\underline{P}(\underline{X}_2 \in B)\\
&\quad = (1 - \underline{P}(\underline{X}_1 \in \bar{A}))(1 - \underline{P}(\underline{X}_2 \in \bar{B}))\\
&\quad = 1 - \underline{P}(\underline{X}_1 \in \bar{A}) - \underline{P}(\underline{X}_2 \in \bar{B}) + \underline{P}(\underline{X}_1 \in \bar{A})\underline{P}(\underline{X}_2 \in \bar{B})
\end{aligned} \tag{2.8}
$$

Since, in general,

$$\underline{P}(\underline{X}_1 \in A, \underline{X}_2 \in B) = 1 - \underline{P}(\underline{X}_1 \in \bar{A}) - \underline{P}(\underline{X}_2 \in \bar{B}) + \underline{P}(\underline{X}_1 \in \bar{A}, \underline{X}_2 \in \bar{B})$$

(2.8) yields $\underline{P}(\underline{X}_1 \in \bar{A}, \underline{X}_2 \in \bar{B}) \le \underline{P}(\underline{X}_1 \in \bar{A})\ \underline{P}(\underline{X}_2 \in \bar{B})$.

The converse is proved in the same way as above. Since (2.8) holds for every k($k = 1,\ldots,n-1$) $\underline{X}$ is $ND(\mathcal{A}^{(n)})$ if and only if $\underline{X}$ is $ND(\bar{\mathcal{A}}^{(n)})$. †

PROPOSION 2.8. *The random vector* $\underline{X}$ *is* $ND(\mathcal{A}^{(k)}, \mathcal{A}^{(n-k)})$ *if and only if* $-\underline{X}$ *is* $ND(-\mathcal{A}^{(k)}, -\mathcal{A}^{(n-k)})$ *and* $\underline{X}$ *is* $ND(\mathcal{A}^{(n)})$ *if and only if* $-\underline{X}$ *is* $ND(-\mathcal{A}^{(n)})$.

REMARK 2.9. From Propositions 2.7 and 2.8 it follows that if for every k $\bar{\mathcal{A}}^{(k)}$ equals $\mathcal{A}^{(k)}$ then $\underline{X}$ is $ND(\mathcal{A}^{(n)})$ if and only if $-\underline{X}$ is $ND(-\mathcal{A}^{(n)})$.

3. Concepts of Negative Dependence

The following collections of upper sets in $R^{(k)}$ will be appeared in most of the following discussion (See Shaked(1982)).

(1) Let $\mathcal{A}_1^{(k)}$ be the collection of all open upper orthants in R^n, that is, $A \in \mathcal{A}_1^{(k)}$ if and only if

$$A = \{x : x_i > a_i,\ i = 1, \ldots, n\} \tag{3.1}$$

for some $a_i \in [-\infty, \infty], i = 1, \ldots, n$.

(2) Let $\mathcal{A}_2^{(k)}$ be the collection of all open upper half spaces, that is, $A \in \mathcal{A}_2^{(k)}$ if and only if

$$A = \{x : \sum_{i=1}^{n} a_i x_i > a_0\} \tag{3.2}$$

for some $a_0 \in [-\infty, \infty]$ *and* $a_i \in [0, \infty], i = 1, \ldots, n$.

(3) Let $\mathcal{A}_3^{(k)}$ be the collection of all sets of the form

$$A = \bigcap_{1 \le \beta \le \gamma} \bigcup_{\alpha \in C_\beta} \{x : x_\alpha > a_\alpha\} \tag{3.3.i}$$

for some $a_i \in [-\infty, \infty], i = 1, \cdots, n$, or of the form

$$A = \bigcup_{1 \le \beta \le \delta} \bigcap_{\alpha \in P_\beta} \{x : x_\alpha > a_\alpha\} \tag{3.3.ii}$$

for some $a_i \in [-\infty, \infty], i = 1, \cdots, n$, where, for some positive integers γ and δ, $C_\beta \subset \{1, \ldots, n\}$, $\beta = 1, \ldots, \gamma$ and $P_\beta \subset \{1, \ldots, n\}$, $\beta = 1, \ldots, \delta$.

(4) Let $\mathcal{A}_4^{(k)}$ be the collection of all convex open upper sets in R^k.

(5) Let $\mathcal{A}_5^{(k)}$ be the collection of all open upper sets in R^k.

REMARK 3.1. (1). From Proposition 2.4 it follows that random variables $X_1, \ldots, X_n$ are negatively associated if and only if $X_1, \ldots, X_n$ is $\mathrm{ND}(\mathcal{A}_5^{(n)})$.
(2). From Proposition 2.6 it follows that

$$\begin{array}{ccccc} & & ND(\mathcal{A}_5^{(n)}) & & \\ & \swarrow & & \searrow & \\ & ND(\mathcal{A}_4^{(n)}) & & & ND(\mathcal{A}_3^{(n)}) \\ \swarrow & & \searrow & & \swarrow \\ ND(\mathcal{A}_2^{(n)}) & & & ND(\mathcal{A}_1^{(n)}) & \end{array}$$

Some of the results of Section 2 can be specialized now to the notions of this section as follows. Since for j = 2,3,5, $\bar{\mathcal{A}}_j^{(n)} = -\mathcal{A}_j^{(n)}$ Theorem 3.1 is obtained from the Remark 2.9 following Proposition 2.8

THEOREM 3.1. *For j = 2,3,5,* $\underline{X}$ *is* $ND(\mathcal{A}_j^{(n)})$ *if and only if* $-\underline{X}$ *is* $ND(\mathcal{A}_j^{(n)})$.

THEOREM 3.2. *(a) For j = 1,3,4,5, if* $\underline{X}$ *is* $ND(\mathcal{A}_j^{(n)})$ *then* $\underline{X}$ *is NUOD.*
(b) For j = 1,3,4,5, if $\underline{X}$ *is* $ND(-\mathcal{A}_j^{(n)})$ *then* $\underline{X}$ *is NLOD.*

Proof. (a). By Remark 3.1, preceding Theorem 3.1 it is enough to prove (a) for j = 1. Let $\underline{X} = (X_1,\ldots,X_n)$. Note that $\underline{X}$ is $\mathrm{ND}(\mathcal{A}_1^{(k)},\mathcal{A}_1^{(n-k)})$ for every k (k = 1,...,n-1):

When k = 1 take $\underline{X}_1 = (X_1)$ and $\underline{X}_2 = (X_2,\ldots,X_n)$ as a partition of $\underline{X}$ and take $\underline{a}_1 = (a_1)$ and $\underline{a}_2 = (a_2,\ldots,a_n)$ as a partition of $\underline{a}$, then

$$\begin{aligned} &\underline{P}(X_1 > a_1, \ldots, X_n > a_n) \\ &\quad = \underline{P}(\underline{X} > \underline{a}) = \underline{P}(\underline{X}_1 > \underline{a}_1, \underline{X}_2 > \underline{a}_2) \le \underline{P}(\underline{X}_1 > \underline{a}_1)\underline{P}(\underline{X}_2 > \underline{a}_2) \\ &\quad = \underline{P}(X_1 > a_1)\underline{P}(X_2 > a_2, \ldots, X_n > a_n) \end{aligned}$$

When k = 2 take $\underline{X}_1 = (X_1, X_2)$ and $\underline{X}_2 = (X_3,\ldots,X_n)$ as a partition of $\underline{X}$ and take $\underline{a}_1 = (a_1,a_2)$ and $\underline{a}_2 = (a_3,\ldots,a_n)$ as a partition of $\underline{a}$, then

$$\begin{aligned} &\underline{P}(X_1 > a_1, \ldots, X_n > a_n) \\ &\quad = \underline{P}(\underline{X}_1 > \underline{a}_1, \underline{X}_2 > \underline{a}_2) \\ &\quad \le \underline{P}(\underline{X}_1 > \underline{a}_1)\underline{P}(\underline{X}_2 > \underline{a}_2) \\ &\quad = \underline{P}(X_1 > a_1, X_2 > a_2)\underline{P}(X_3 > a_3, \ldots, X_n > a_n). \end{aligned} \tag{3.4}$$

By choosing $a_1 = -\infty$ in (3.4) we obtain

$$\underset{\sim}{P}(X_2 > a_2, \ldots, X_n > a_n) \leq \underset{\sim}{P}(X_2 > a_2)\underset{\sim}{P}(X_3 > a_3, \ldots, X_n > a_n).$$

We proceed by induction and finally,

When $k = n\text{-}1$ take $\underset{\sim}{X}_1 = (X_1, \ldots, X_{n-1})$ and $\underset{\sim}{X}_2 = (X_n)$ as a partition of $\underset{\sim}{X}$ and take $\underset{\sim}{a}_1 = (a_1, \ldots, a_{n-1})$ and $\underset{\sim}{a}_2 = (a_n)$ as a partition of $\underset{\sim}{a}$, then

(3.5)

$$\begin{aligned}\underset{\sim}{P}(X_1 > a_1, \ldots, X_n > a_n) &= \underset{\sim}{P}(\underset{\sim}{X}_1 > \underset{\sim}{a}_1, \underset{\sim}{X}_2 > \underset{\sim}{a}_2)\\ &\leq \underset{\sim}{P}(\underset{\sim}{X}_1 > \underset{\sim}{a}_1)\underset{\sim}{P}(\underset{\sim}{X}_2 > \underset{\sim}{a}_2)\\ &= \underset{\sim}{P}(X_1 > a_1, \ldots, X_{n-1} > a_{n-1})\underset{\sim}{P}(X_n > a_n)\end{aligned}$$

By choosing $a_1 = -\infty, \ldots, a_{n-2} = -\infty$ in (3.5) we obtain

$$\underset{\sim}{P}(X_{n-1} > a_{n-1}, \ldots X_n > a_n) \leq \underset{\sim}{P}(X_{n-1} > a_{n-1})\underset{\sim}{P}(X_n > a_n).$$

Producting the above inequalities side by side and cancelling the common terms we obtain

$$\underset{\sim}{P}(X_1 > a_1, \ldots, X_n > a_n) \leq \prod_{i=1}^{n} \underset{\sim}{P}(X_i > a_i)$$

(b). Since for $j = 3,4,5$, $-\mathcal{A}_j \supset -\mathcal{A}_1$ it is enough to prove (b) for $j = 1$:

$$\begin{aligned}\underset{\sim}{X} \text{ is } ND(-\mathcal{A}_1) &\Longrightarrow -\underset{\sim}{X} \text{ is } ND(\mathcal{A}_1)\\ &\Longrightarrow -\underset{\sim}{X} \text{ is } NUOD \text{ by (a)}\\ &\Longrightarrow -\underset{\sim}{X} \text{ is } NLOD.\end{aligned}$$

REMARK 3.2. Theorem 3.2 indicates that for $j = 1,3,4$, $ND(\mathcal{A}_j^{(n)})$ notions are weaker than negative association and stronger than the orthant dependence notion.

THEOREM 3.3. *For $j = 3,4,5$, (a). if $\underset{\sim}{X}$ is $ND(\mathcal{A}_j^{(n)})$ then $\underset{\sim}{X}$ is NLOD. (b). if $\underset{\sim}{X}$ is $ND(-\mathcal{A}_j^{(n)})$ then $\underset{\sim}{X}$ is NUOD.*

Proof. The proof is simlar to the proof of (c), (d) of Theorem 3.3 in Shaked(1982) †

Theorems 3.2 and 3.3 do not say anything about the $ND(\mathcal{A}_2^{(n)})$ family. However, we have the following theorem from Definition 2.3 and (3.2).

THEOREM 3.4. *$\underset{\sim}{X}$ is ND($\mathcal{A}_2^{(n)}$) if and only if $\underset{\sim}{X}$ is LNQD.*

THEOREM 3.5. *For $j = 1,\ldots,5$, if $\underset{\sim}{X}$ is ND($\mathcal{A}_j^{(n)}$) then $(X_{\alpha_1}, \ldots, X_{\alpha_m})$ is ND($\mathcal{A}_j^{(m)}$) whenever $\{ \alpha_1, \ldots, \alpha_m \} \subset \{ 1, \ldots, n \}$.*

Proof. For j = 1,3,4,5, if for every set A $\in \mathcal{A}_j^{(n)}$ and for every subset $\{ \alpha_1, \ldots, \alpha_m \} \subset \{ 1,\ldots,n \}$ (m < n), then the set $\{(X_{\alpha_1}, \ldots, X_{\alpha_m}) : (X_1, \ldots, X_n) \in A \}$ belongs to $\mathcal{A}_j^{(m)}$. Also it can be seen that by setting the appropriate a_i in (3.2) equal to zero the above property holds for j = 2. Thus for j = 1,...,5, if $\underset{\sim}{X}$ is ND($\mathcal{A}_j^{(n)}$) then every subvector ($X_{\alpha_1}, \ldots, X_{\alpha_m}$) of $\underset{\sim}{X}$ is ND($\mathcal{A}_j^{(m)}$), where $\{ \alpha_1, \ldots, \alpha_m \} \subset \{ 1,\ldots, n \}$. †

COROLLARY 3.6. *Let $\underset{\sim}{X} = (X_1,\ldots,X_n)$ be an ND($\mathcal{A}_j^{(n)}$) n-variate random vector for $j = 1,3,4,5$, then for every A_1, A_2 disjoint subsets of $\{ 1,\ldots,n \}$, and $x_1,\ldots, x_n$ real*

$\underset{\sim}{P}(X_i > x_i, i = 1, \ldots, n) \leq \underset{\sim}{P}(X_i > x_i, i \in A_1) \underset{\sim}{P}(X_j > x_j, j \in A_2)$.

REMARK 3.3. (1). When n = 2 it follows that NQD $\Leftrightarrow$ ND($\mathcal{A}_5^{(n)}$), ND($\mathcal{A}_2^{(n)}$) $\Leftrightarrow$ ND($\mathcal{A}_1^{(n)}$) and hence for j = 1,...,5, ND($\mathcal{A}_j^{(n)}$) are equivalent.

(2). By combining Theorem 3.4 and the proof of Theorem 10 of Newman (1984) it can be shown that if $\underset{\sim}{X} = (X_1,\ldots,X_n)$ are ND($\mathcal{A}_2^{(n)}$) finite variance random vector with joint and marginal charcteristic functions, ϕ and ϕ_j, j = 1,...,n ; then

$$(3.6) \quad |\phi(r_1, \ldots, r_n) - \prod_{j=1}^{n} \phi_j(r_j)| \leq \frac{1}{2} \sum_{1 \leq j \neq} \sum_{k \leq n} |r_j||r_k| \operatorname{Cov}(X_j, X_k).$$

From Remark 3.1 it follows then that for j = 2,4,5, if $X_1,\ldots,X_n$ is ND($\mathcal{A}_j^{(n)}$) and if the X's are uncorrelated then $X_1,\ldots,X_n$ are jointly independent.

4. Concepts of Functional Negative Dependence

In many instance(see Section 3 of Shaked(1982)) if a random vector $\underline{X}$ is PD($\mathcal{A}, \mathcal{B}$) then there exist two families of real functions $\mathcal{F}$ and $\mathcal{G}$ such that

$$(4.1) \qquad \mathrm{Cov}(f(\underline{X}), g(\underline{X})) \geq 0 \text{ whenever } f \in \mathcal{F}, g \in \mathcal{G}$$

provided the expectations exist. When $\underline{X}$ satisfies (4.1) Shaked(1982) said that $\underline{X}$ is functionally positive dependent relative to $\mathcal{F}$ and $\mathcal{G}$ (denoted by FPD($\mathcal{F}, \mathcal{G}$)). Motivated by (4.1) we introduce the functional negative dependence as follows: Let $\underline{X}_1 = (X_{\pi(1)}, \ldots, X_{\pi(k)})$ and $\underline{X}_2 = (X_{\pi(k+1)}, \ldots, X_{\pi(n)})$ be any partition of $\underline{X} = (X_1, \ldots, X_n)$, $1 \leq$ k $\leq$ n-1, and π be any permutation of { 1,2,...,n }. In many instance, if $\underline{X}$ is ND($\mathcal{A}^{(n)}$, $\mathcal{A}^{(n-k)}$) then there exist a family of real k-variate functions $\mathcal{F}^{(k)}$ and a family of real n−k variate function $\mathcal{F}^{(n-k)}$ such that

$$(4.2) \qquad \mathrm{Cov}(f(\underline{X}_1), g(\underline{X}_2)) \leq 0 \text{ whenever } f \in \mathcal{F}^{(k)}, g \in \mathcal{F}^{(n-k)}$$

provided the expectations exists.

DEFINITION 4.1. If the random vector $\underline{X}$ satisfies (4.2) we say that $\underline{X}$ is functionally negative dependent relative to $\mathcal{F}^{(k)}$ and $\mathcal{F}^{(n-k)}$ (denoted by FND($\mathcal{F}^{(k)}$, $\mathcal{F}^{(n-k)}$)) and if $\underline{X}$ satisfies (4.2) for every k(k = 1,... ,n-1) $\underline{X}$ then is functionally negative dependent relative to $\mathcal{F}^{(n)}$ (denoted by FND($\mathcal{F}^{(n)}$)).

PROPOSITION 4.2. *If $\mathcal{F}^{(k)} \subset \tilde{\mathcal{F}}^{(k)}$ and $\mathcal{F}^{(n-k)} \subset \tilde{\mathcal{F}}^{(n-k)}$ then FND ($\tilde{\mathcal{F}}^{(k)}$, $\tilde{\mathcal{F}}^{(n-k)}$) implies FND ($\mathcal{F}^{(k)}$,$\mathcal{F}^{(n-k)}$) and if $\mathcal{F}^{(k)} \subset \tilde{\mathcal{F}}^{(k)}$, k=1,...,n-1, then FND($\tilde{\mathcal{F}}^{(n)}$) implies FND ($\mathcal{F}^{(n)}$).*

Consider now the following collections of increasing functions in R^n or in $R^n_+ = \{x : x \geq 0\}$ which will be appeared in most of the following discussion (see Shaked(1982)).

(1) Let $\mathcal{F}_1^{(n)}$ be the collection of all functions, defined on R^n_+, of the form

$$(4.2) \qquad f(x) = \min_{1 \leq i \leq n} \{b_i x_i\} \text{ for some } b_i \in [0, \infty],$$

(2) Let $\mathcal{F}_2^{(n)}$ be the collection of all funcions, defined on R_+^n, of the form

$$(4.3) \qquad f(x) = \sum_{1=1}^{n} a_i x_i \text{ for some } a_i \in [0, \infty], i = 1 \ldots, n$$

(3) Let $\mathcal{F}_3^{(n)}$ be the collection of all functions, defined on R_+^n, of the form

$$(4.4.\text{i}) \quad f(x) = \min_{1 \le \beta \le \gamma} \max_{\alpha \in C_\beta} b_\alpha x_\alpha \quad \text{for some } b_i \in [0, \infty], i = 1, \cdots, n$$

or of the form

$$(4.4.\text{ii}) \qquad f(x) = \max_{1 \le \beta \le \delta} \min_{\alpha \in P_\beta} b_\alpha x_\alpha \text{ for some } b_i \in [0, \infty], i = 1, \cdots, n$$

where, for some positive integer γ and δ, $C_\beta \in \{ 1,\ldots,n \}$, $= 1,\ldots,\gamma$ and $P_\beta \in \{1, \ldots, n \}$, $\beta = 1,\ldots,\delta$.

(4) Let $\mathcal{F}_4^{(n)}$ be the collection of all concave increasing functions on R^n (or on R_+^n when we deal with nonnegative random vectors).

(5) Let $\mathcal{F}_5^{(n)}$ be the collection of all measurable increasing functions on R^n (or on R_+^n when we deal with nonnegative random vectors).

In the following the superscript n on the $\mathcal{F}$'s will be omitted when there is no danger of a confusion.

REMARK. (1). From (5) and Definition 4.1 it follows that $\underset{\sim}{X} = (X_1,\ldots,X_n)$ is negatively associated(NA) if and only if $\underset{\sim}{X}$ is FND($\mathcal{F}_5^{(n)}$).

(2). It follows that for j = 1,...,5, if $\underset{\sim}{X}$ is FND($\mathcal{F}_j^{(n)}$) then every m-variate subvectors of $\underset{\sim}{X}$ (m < n) is FND($\mathcal{F}_j^{(m)}$).

(3). From Proposition 4.2 it follows that

$$\begin{array}{ccccccc} & & & FND(\mathcal{F}_5^{(n)}) & & & \\ & & \swarrow & & \searrow & & \\ & FND(\mathcal{F}_4^{(n)}) & & & & FND(\mathcal{F}_3^{(n)}) & \\ \swarrow & & \searrow & & \swarrow & & \\ FND(\mathcal{F}_2^{(n)}) & & & FND(\mathcal{F}_1^{(n)}) & & & \end{array}$$

Now we are going to show that for j = 1,...,5, the notion of ND($\mathcal{A}_j^{(n)}$) essentially implies the notion of FND($\mathcal{F}_j^{(n)}$). First the following Lemma which characterizes ND($\mathcal{A}_j^{(n)}$) is proven.

LEMMA 4.3. *For $j = 1,\ldots,5$, $\underline{X}$ is $ND(\mathcal{A}_j^{(n)})$ if and only if for arbitrary partitions $\underline{X}_1 = (X_{\pi(1)},\ldots,X_{\pi(k)})$, $\underline{X}_2 = (X_{\pi(k+1)},\ldots,X_{\pi(n)})$ of $\underline{X}$, π be any permutation of $\{1,\ldots,n\}$ and every k ($k = 1,\ldots,n-1$)*

$$P(f(\underline{X}_1) > s, g(\underline{X}_2) > t) \le P(f(\underline{X}_1) > s)P(g(\underline{X}_2) > t) \tag{4.6}$$

whenever $f\in \mathcal{F}_j^{(k)}$, $g\in \mathcal{F}_j^{(n-k)}$ provided $\underline{X}$ is nonnegative. Without the nonnegativity assumption, (4.6) holds for $j = 2,4,5$.

Proof. When $j = 1$ and $\underline{X} \ge 0$ then the x's in (3.1) are nonnegative and for every $k(k = 1,\ldots,n-1)$ $A \in \mathcal{A}_1^{(k)}$, $B \in \mathcal{A}_1^{(n-k)}$ if and only if $A = \{\underline{X}_1 : \min_{1\le j\le k} b_j X_{\pi(j)} > s\}$ for some $s \in [-\infty, \infty]$ and $b_j \ge 0$, $j = 1,\ldots,k$ and $B = \{\underline{X}_2 : \min_{k+1\le j\le n} c_j X_{\pi(j)} > t\}$ for some $t \in [-\infty, \infty]$ and $c_j \ge 0$, $j = k+1,\ldots,n$. Let $f(\underline{X}_1) = \min_{1\le j\le k} \{b_j X_{\pi(j)}\}$ for some $b_j \in [0,\infty]$ and $g(\underline{X}_2) = \min_{k+1\le j\le n} \{c_j X_{\pi(j)}\}$ for some $c_j \in [0,\infty]$. Then from definition of $ND(\mathcal{A}_1^{(n)})$ for every k($k = 1,\ldots,n-1$) $P(f(\underline{X}_1) > s, g(\underline{X}_2) > t) = P(\underline{X}_1 \in A, \underline{X}_2 \in B) \le P(\underline{X}_1 \in A)\, P(\underline{X}_2 \in B) = P(f(\underline{X}_1) \ge s)\, P(g(\underline{X}_2) \ge t)$ for $f \in \mathcal{F}_1^{(k)}$, $g \in \mathcal{F}_1^{(n-k)}$ thus (4.6) holds. When $j = 2$ then (4.6) follows directly from the definition of $ND(\mathcal{A}_2^{(n)})$, (see (3.2)). When $j = 3$ and $\underline{X} \ge 0$ then, to construct sets in $\mathcal{A}_3^{(k)}$ and $\mathcal{A}_3^{(n-k)}$ we consider in (3.3.i) or in (3.3.ii) only the sets for every k

$$\{\underline{X}_1 : b_j X_{\pi(j)} > 1\} \tag{4.7}$$

for some $j \in \{1,\ldots,k\}$ and $b_1 \in [0, \infty]$, and

$$\{\underline{X}_2 : c_j X_{\pi(j)} > 1\} \tag{4.8}$$

for some $j \in \{k+1,\ldots,n\}$ and $c_j \in [0,\infty]$.

By taking unions and intersections of sets of the form (4.4) and (4.5), respectively. We obtain sets of the form

$$\{\underline{X}_1 : f(\underline{X}_1) > 1\} \tag{4.9}$$

for some $f$of the form (2.12) or (2.13) and

$$\{\underline{X}_2 : g(\underline{X}_2) > 1\} \tag{4.10}$$

for some $g$of the form (2.12) or (2.13) that is, A $\in \mathcal{A}_3^{(k)}$ and B $\in \mathcal{A}_3^{(n-k)}$ if and only if A is of the form (4.9) and B is of the form (4.10). Using the homogeneity and nonnegativity of (4.9) and (4.10), finally we observe for every k(k = 1,...,n-1) A $\in \mathcal{A}_3^{(k)}$ and B $\in \mathcal{A}_3^{(n-k)}$ if and only if A = { $\underline{X}_1$: f($\underline{X}_1$) > b } for some f $\in \mathcal{F}_3^{(k)}$ and some b $\in [0,\infty]$ and B = { $\underline{X}_2$: g($\underline{X}_2$) > c } for some g $\in \mathcal{F}_3^{(n-k)}$ and some c $\in [0,\infty]$. Thus by the definition of ND($\mathcal{A}_3^{(n)}$) for every k(k = 1,...,n-1) $\underline{P}$($\underline{X}_1 \in$ A, $\underline{X}_2 \in$ B) $\leq \underline{P}$($\underline{X}_1 \in$ A) $\underline{P}$($\underline{X}_2 \in$ B) if and only if $\underline{P}$($f(\underline{X}_1) >$ b, $g(\underline{X}_2) >$ c) $\leq \underline{P}$($f(\underline{X}_1) >$ b) $\underline{P}$($g(\underline{X}_2) >$ c) for $f \in \mathcal{F}_3^{(k)}$, $g \in \mathcal{F}_3^{(n-k)}$.

When j = 4, first assume that X satisfied (4.6). Let π be any permutation of { 1,...,n }, $\underline{X}_1$ = ($X_{\pi(1)}, \ldots, X_{\pi(k)}$) and $\underline{X}_2$ = ($X_{\pi(k+1)}, \ldots, X_{\pi(n)}$) be arbitrary partitions of $\underline{X}$ and A and B in $\mathcal{A}_4^{(k)}$ and $\mathcal{A}_4^{(n-k)}$, respectively. Since A and B are convex, open and upper sets they can be approximated by intersections of sets of the form

$$\{\underline{X}_1 : \sum_{j=1}^{k} \mathrm{a}_j X_{\pi(j)} > 1\}, \text{ where } a_j \geq 0, j = 1, \ldots, k,$$

and

$$\{\underline{X}_2 : \sum_{j=k+1}^{n} \mathrm{b}_j X_{\pi(j)} > 1\}, \text{ where } \mathrm{b}_j \geq 0, j = 1, \ldots, n,$$

Explicitly, for every $\varepsilon > 0$ there exist K_1 and K_2 such that

$$|\underline{P}(\underline{X}_1 \in A) - \underline{P}(\min_{1 \leq l \leq K_1} \sum_{i=1}^{k} \mathrm{a}_i^{(l)} X_{\pi(i)} > 1)| < \varepsilon$$

where $\mathrm{a}_i^{(l)} \geq 0$, i = 1,...,k ; l = 1,...,K_1 and

$$|\underline{P}(\underline{X}_2 \in B) - \underline{P}(\min_{1 \leq l \leq K_2} \sum_{j=k+1}^{n} \mathrm{b}_j^{(l)} X_{\pi(j)} > 1)| < \varepsilon$$

where $\mathrm{b}_j^{(l)} \geq 0$, j = k+1, ..., n ; l = 1, ..., K_2 Denoting $f_{k_1}(\underline{X}_1) = \min_{1 \leq l \leq K_1} \sum_{i=1}^{k} \mathrm{a}_i^{(l)} X_{\pi(i)}$ and $g_{k_2}(\underline{X}_2) = \min_{1 \leq l \leq K_2} \sum_{j=k+1}^{k} \mathrm{b}_j^{(l)} X_{\pi(j)}$, We can also assume that

$$|\underset{\sim}{P}(\underset{\sim}{X}_1 \in A, \underset{\sim}{X}_2 \in B) - \underset{\sim}{P}(f_{k_1}(\underset{\sim}{X}_1) > 1, g_{k_2}(\underset{\sim}{X}_2) > 1)| < \varepsilon.$$

Since $f_{k_1} \in \mathcal{F}_4^{(k)}$ and $g_{k_2} \in \mathcal{F}_4^{(n-k)}$ for every k(k = 1,...,n-1), thus it follows from (4.6) that

$$\begin{aligned}\underset{\sim}{P}(\underset{\sim}{X}_1 \in A, \underset{\sim}{X}_2 \in B) - \varepsilon &\le \underset{\sim}{P}(f_{k_1}(\underset{\sim}{X}_1) > 1, g_{k_2}(\underset{\sim}{X}_2) > 1) \\ &\le [\underset{\sim}{P}(f_{k_1}(\underset{\sim}{X}_1) > 1)\underset{\sim}{P}(g_{k_2}(\underset{\sim}{X}_2) > 1) \\ &\le [\underset{\sim}{P}(\underset{\sim}{X}_1 \in A) + \varepsilon][\underset{\sim}{P}(\underset{\sim}{X}_2 \in B) + \varepsilon].\end{aligned}$$

Letting $\varepsilon \to 0$ we obtain

$$\underset{\sim}{P}(\underset{\sim}{X}_1 \in A, \underset{\sim}{X}_2 \in B) \le \underset{\sim}{P}(\underset{\sim}{X}_1 \in A)\underset{\sim}{P}(\underset{\sim}{X}_2 \in B),$$

that is, $\underset{\sim}{X}$ is ND($\mathcal{A}_4^{(k)}$, $\mathcal{A}_4^{(n-k)}$) for every k(k = 1,...,n-1) and hence $\underset{\sim}{X}$ is ND($\mathcal{A}_4^{(n)}$).

To show the converse assume that X is ND($\mathcal{A}_4^{(n)}$), let $f \in \mathcal{A}_4^{(k)}$ and $g \in \mathcal{A}_4^{(n-k)}$ for every k(k = 1,...,n-1). Then for every a and b the set A = { $\underset{\sim}{X}_1$: $f(\underset{\sim}{X}_1) >$ a } is in $\mathcal{A}_4^{(k)}$ and B = { $\underset{\sim}{X}_2$: $g(\underset{\sim}{X}_2) >$ b } is in $\mathcal{A}_4^{(n-k)}$. Thus since $\underset{\sim}{X}$ is ND($\mathcal{A}_4^{(n)}$),

$$\begin{aligned}\underset{\sim}{P}(f(\underset{\sim}{X}_1) > \mathrm{a}, g(\underset{\sim}{X}_2) > \mathrm{b}) &= \underset{\sim}{P}(\underset{\sim}{X}_1 \in A, \underset{\sim}{X}_2 \in B) \\ &\le \underset{\sim}{P}(\underset{\sim}{X}_1 \in A)\underset{\sim}{P}(X_2 \in B) \\ &= \underset{\sim}{P}(f(\underset{\sim}{X}_1) > \mathrm{a})\underset{\sim}{P}(g(\underset{\sim}{X}_2) > \mathrm{b})\end{aligned}$$

that is, (4.6) holds.

THEOREM 4.4. *For j = 1,...,5, if the random vector $\underset{\sim}{X}$ is ND($\mathcal{A}_j^{(n)}$) and nonnegative then $\underset{\sim}{X}$ is FND($\mathcal{F}_j^{(n)}$). If it is not assumed that $\underset{\sim}{X}$ is nonnegative then the above is true for j = 2,4,5.*

Proof. $\underset{\sim}{X}$ is ND($\mathcal{A}_j^{(n)}$) then, by Lemma 4.3 for every pair of partitions $\underset{\sim}{X}_1$ = (X_{α_1} ,...,X_{α_k}), $\underset{\sim}{X}_2$ = ($X_{\alpha_{k+1}}$,...,X_{α_n}) of $\underset{\sim}{X}$ Cov($f(\underset{\sim}{X}_1), g(\underset{\sim}{X}_2)$) $\le$ 0 whenever $f \in \mathcal{F}_j^{(k)}$, $g \in \mathcal{F}_j^{(n-k)}$, $1 \le$ k $\le$ n-1, and { $\alpha_1,\ldots,\alpha_k$ } $\subset$ { 1,... ,n }, { $\alpha_{k+1},\ldots,\alpha_n$ } $\subset$ { 1,...,n }, that is, $\underset{\sim}{X}$ is FND($\mathcal{F}_j^{(n)}$). †

THEOREM 4.5. *For $j = 1,2,5$, if $\underset{\sim}{X} = (X_1, \ldots, X_n)$ and $\underset{\sim}{Y} = (Y_1, \ldots, Y_m)$ are nonnegative independent random vectors which are ND($\mathcal{A}_j^{(n)}$) and ND($\mathcal{A}_j^{(m)}$), respectively, then ($\underset{\sim}{X}$, $\underset{\sim}{Y}$) is ND($\mathcal{A}_j^{(n+m)}$). Without the nonnegativity assumption, the above is true for $j = 2,5$.*

Proof. The results is well known when j = 5 (NA) from Property P_7 of Joag - Dev and Proŝchan (1983). Let ($\underset{\sim}{X}_1$, $\underset{\sim}{X}_2$) and ($\underset{\sim}{Y}_1$, $\underset{\sim}{Y}_2$) denote arbitrary partitions of $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ respectively. Put $\underset{\sim}{X}_1 = (X_{\alpha(1)}, \ldots, X_{\alpha(k)})$, $\underset{\sim}{X}_2 = (X_{\alpha(k+1)}, \ldots, X_{\alpha(n)})$, $\underset{\sim}{Y}_1 = (Y_{\beta(1)}, \ldots, Y_{\beta(r)})$ and $\underset{\sim}{Y}_2 = (Y_{\beta(r+1)}, \ldots, Y_{\beta(m)})$. where α, β are arbitrary permutations of {1,...,n} and {1,...,m}, respectively. By Lemma 4.3 it is sufficient to show that for j = 1,2,

$$\text{(4.11)} \qquad \mathrm{Cov}(h_1(f(\underset{\sim}{X}_1, \underset{\sim}{Y}_1)), h_2(g(\underset{\sim}{X}_2, \underset{\sim}{Y}_2))) \leq 0$$

whenever $f \in \mathcal{F}_j^{(k+r)}$, $g \in \mathcal{F}_j^{(n+m-k-r)}$
and h_1 and h_2 are increasing functions.

Denote by G the distribution function of $\underset{\sim}{y}_1$, $\underset{\sim}{y}_2$ and let f, g, h_1 and h_2 be as in (4.8). Note that j = 1,2, and for every $\underset{\sim}{Y}_1$, $\underset{\sim}{Y}_2$, $h_1(f(\cdot, \underset{\sim}{y}_1))$ and $h_2(g(\cdot, \underset{\sim}{y}_2))$ are increasing functions of functions in $\mathcal{F}_j^{(r)}$, $\mathcal{F}_j^{(m-r)}$ respectively, thus, by Lemma 4.3

$$\begin{aligned} &E[h_1(f(\underset{\sim}{X}_1, \underset{\sim}{Y}_1))h_2(g(\underset{\sim}{X}_2, \underset{\sim}{Y}_2))] \\ &\quad = \int_{R^m} E[h_1(f(\underset{\sim}{X}_1, \underset{\sim}{y}_1))h_2(g(\underset{\sim}{X}_2, \underset{\sim}{y}_2)]dG(\underset{\sim}{y}) \\ &\quad \leq \int_{R^m} E[h_1(f(\underset{\sim}{X}_1, \underset{\sim}{y}_2))h_2(g(\underset{\sim}{X}_2, \underset{\sim}{y}_2)]dG(\underset{\sim}{y}) = \ (I) \quad \text{(say)} \end{aligned}$$

Now $\psi_1(\cdot) = E[h_1(f(\underset{\sim}{X}_1, \cdot))]$ and $\psi_2(\cdot) = E[h_2(g(\underset{\sim}{X}_2, \cdot))]$ are also increasing functions of functions in $\mathcal{F}_j^{(k)}$, $\mathcal{F}_j^{(n-k)}$ respectively, thus, again by Lemma 4.3,

$$\begin{aligned} (I) &= \int_{R^m} \psi_1(\underset{\sim}{y}_1)\psi_2(\underset{\sim}{y}_2)dG(\underset{\sim}{y}) = E[\psi_1(\underset{\sim}{Y}_1)\psi_2(Y_2)] \\ &\leq E(\psi_1(\underset{\sim}{Y}_1))E(\psi_2(\underset{\sim}{Y}_2)) \\ &= E[h_1(f(\underset{\sim}{X}_1, \underset{\sim}{Y}_1))]E[h_2(g(\underset{\sim}{X}_2, \underset{\sim}{Y}_2))]. \end{aligned}$$

So

$$\mathrm{Cov}(h_1(f(\underset{\sim}{X}_1, \underset{\sim}{Y}_1)), h_2(g(\underset{\sim}{X}_2, \underset{\sim}{Y}_2))) \leq 0,$$

Hence, by Lemma 4.3, ($\underset{\sim}{X}$, $\underset{\sim}{Y}$) is ND($\mathcal{A}_j^{(n+m)}$)). †

THEOREM 4.6. *For $j = 1, \ldots, 5$, if $\underset{\sim}{X} = (X_1, \ldots, X_n)$ is FND($\mathcal{F}_j^{(n)}$) then $\underset{\sim}{Y} = (f_1(\underset{\sim}{X}_1), f_2(\underset{\sim}{X}_2), \ldots, f_k(\underset{\sim}{X}_k))$ is FND($\mathcal{F}_j^{(k)}$) whenever ($\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_k$) is arbitrary partitions of $\underset{\sim}{X}$ and $f_j \in \mathcal{F}_j^{(r_j)}$, provided r_j is the number of components of $\underset{\sim}{X}_j$ and $\sum_{j=1}^{k} r_j = n$.*

ACKNOWLEDGMENTS.. The authors wish to thank the referee for a very thorough review of this paper. Part of this work appears in the Wonkwang University doctoral dissertation of the second author, who gratefully acknowledges numerous helpful discussions with Bong Dai Choi.

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