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OPTIMALITY AND DUALITY FOR MULTIOBJECTIVE ρ-INVEX PROGRAMS

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1. Introduction

M. A. Hanson [3] defined the invex function which is a generalization of the convex function, proved that the Kuhn-Tucker conditions are sufficient for a global minima of the scalar optimization problem concerning with invex functions and established Wolfe duality theorems [12] for this problem. On the other hand, J. P. Vial [10] defined the ρ-convex function and considered various properties of this function. Subsequently R. R. Egudo [2] showed that Wolfe type duality theorems and Mond-Weir type duality theorems hold for the multiobjective optimization problem which consist of ρ-convex functions. V. Jeyakumar [4] introduced the ρ-invex function which is an extension of the invex function and the ρ-convex function, investigated the sufficiency of the Kuhn-Tucker conditions for scalar nonlinear optimization programs which are composed of ρ-invex functions, and considered Wolfe duality theorems for these programs. Recently, D. S. Kim and G. M. Lee [6] proved sufficiency of the Kuhn-Tucker conditions and duality theorems for multiobjective invex programming.

In this paper, we prove that the Kuhn-Tucker conditions are sufficient for an efficient solution of the multiobjective optimization problem which consists of ρ-invex functions, and establish Wolfe type duality theorems and Mond-Weir type duality theorems for this problem.

Throughout this paper, we use the following conventions :

Let R^k be a k-dimensional Euclidean space, $x = (x_1, \cdots, x_k)^t \in R^k$, $y = (y_1, \cdots, y_k)^t \in R^k$ and $R^k_+ = \{z = (z_1, \cdots, z_k)^t \in R^k : z_i \geq 0\}$.

1. $x < y$ if and only if $x_i < y_i$, $i = 1, \cdots, k$.

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2. $x \leq y$ if and only if $x_i \leq y_i,\ i = 1, \cdots, k$.
3. $x \leq y$ if and only if $x_i \leq y_i,\ i = 1, \cdots, k$, but $x \neq y$.
4. $x \not\leq y$ is the negation of $x \leq y$.

DEFINITION 1.1. A differentiable function $h : R^n \to R$ is ρ-invex with respect to functions $\eta, \theta : R^n \times R^n \to R^n$ if and only if there exists some real number ρ such that for each $x, u \in R^n$,

$$h(x) - h(u) \geq \nabla h(u)\eta(x, u) + \rho\|\theta(x, u)\|^2,$$

where $\|\cdot\|$ is the Euclidean norm on R^n.

We consider the multiobjective program :

$$\text{(P)} \qquad \begin{array}{ll} \text{Minimize} & f(x) \\ \text{subject to} & x \in X = \{x \in R^n : g(x) \leq 0\} \end{array}$$

where $f : R^n \to R^p$ and $g : R^n \to R^m$ are differentiable functions.

In relation to (P), we consider the following two multiobjective maximization problems.

The Wolfe vector dual of (P) [12] :

(WVD) Maximize $f(x) + y^t g(x)e$ subject to $(x, \lambda, y) \in Y_W$, where $Y_W = \{(x, \lambda, y) : \nabla\lambda^t f(x) + \nabla y^t g(x) = 0, y \geq 0, \lambda \in \Lambda^+\}, e = (1, \cdots, 1)^t \in R^p$ and $\Lambda^+ = \{\lambda > 0, \lambda^t e = 1\}$.

The Mond-Weir vector dual of (P) [9] :

$(MWVD)$ Maximize $f(x)$ subject to $(x, \lambda, y) \in Y_M$, where $Y_M = \{(x, \lambda, y) : \nabla\lambda^t f(x) + \nabla y^t g(x) = 0, \lambda \in \Lambda^+, y \geq 0, y^t g(x) \geq 0\}$.

In this, optimization in (P), (WVD) and (MWVD) means obtaining efficient solutions for the corresponding programs. Recently, T. Weir [11] considered the above two dual problems of (P) by using the concept of the proper efficiency.

DEFINITION 1.2. (a). A point $\overline{x} \in X$ is an efficient solution for (P) if and only if for any $x \in X$, $f(x) \not\leq f(\overline{x})$.
(b). A point $(\overline{x}, \overline{\lambda}, \overline{y}) \in Y_W$ is an efficient solution for (WVD) if and only if for any $(x, \lambda, y) \in Y_W$, $f(\overline{x}) + \overline{y}^t g(\overline{x})e \not\leq f(x) + y^t g(x)e$.
(c). A point $(\overline{x}, \overline{\lambda}, \overline{y}) \in Y_M$ is an efficient solution for (MWVD) if and only if for any $(x, \lambda, y) \in Y_M$, $f(\overline{x}) \not\leq f(x)$.

2. Optimality Conditions

Now, we consider optimality conditions for an efficient solutions for (P).

THEOREM 2.1. *Suppose that f_i, $i = 1, \cdots, p$ are ρ_i-invex with respect to η and θ and g_j, $j = 1, \cdots, m$ are σ_j-invex with respect to η and θ. If there exist $\overline{\lambda} > 0$, $\overline{\lambda} \in R^p$ and $\overline{y} \in R^m_+$ such that $\nabla\overline{\lambda}^t f(\overline{x}) + \nabla\overline{y}^t g(\overline{x}) = 0$, $\overline{y}^t g(\overline{x}) = 0$ and $g(\overline{x}) \leq 0$, and if, furthermore, $\sum_{i=1}^p \overline{\lambda}_i\rho_i + \sum_{j=1}^m \overline{y}_j\sigma_j \geq 0$, then $\overline{x}$ is an efficient solution of (P).*

Proof. Suppose that $\overline{x}$ is not an efficient solution for (P). Then, there exists an $x \in X$ such that $f(x) \leq f(\overline{x})$. Since $\overline{\lambda} > 0$, we have $\overline{\lambda}^t f(x) < \overline{\lambda}^t f(\overline{x})$. By the $\sum_{i=1}^p \overline{\lambda}_i\rho_i$-invexity of $\overline{\lambda}^t f(\cdot)$,

$$\nabla\overline{\lambda}^t f(\overline{x})\eta(x,\overline{x}) + \sum_{i=1}^p \overline{\lambda}_i\rho_i\|\theta(x,\overline{x})\|^2 < 0. \tag{1}$$

Since $\overline{y}^t g(\overline{x}) = 0$ and $\overline{y}^t g(x) \leq 0$, we have $\overline{y}^t g(x) - \overline{y}^t g(\overline{x}) \leq 0$. By the $\sum_{j=1}^m \overline{y}_j\sigma_j$-invexity of $\overline{y}^t g(\cdot)$,

$$\nabla\overline{y}^t g(\overline{x})\eta(x,\overline{x}) + \sum_{j=1}^m \overline{y}_j\sigma_j\|\theta(x,\overline{x})\|^2 \leq 0. \tag{2}$$

From (1) and (2),

$$[\nabla\overline{\lambda}^t f(\overline{x}) + \nabla\overline{y}^t g(\overline{x})]\eta(x,\overline{x}) + (\sum_{i=1}^p \overline{\lambda}_i\rho_i + \sum_{j=1}^m \overline{y}_j\sigma_j)\|\theta(x,\overline{x})\|^2 < 0.$$

By the assumption $(\sum_{i=1}^p \overline{\lambda}_i\rho_i + \sum_{j=1}^m \overline{y}_j\sigma_j) \geq 0$, we have

$$[\nabla\overline{\lambda}^t f(\overline{x}) + \nabla\overline{y}^t g(\overline{x})]\eta(x,\overline{x}) < 0.$$

This contradicts our hypothesis.

LEMMA 2.1 ([1],[5]). *$\overline{x} \in X$ is an efficient solution for (P) if and only if $\overline{x}$ is a solution for (P_k), where (P_k) is the following problem. Minimize $f_k(x)$ subject to $f_j(x) \leq f_j(\overline{x})$ for all $j \neq k, g(x) \leq 0$, for each $k = 1, \cdots, p$.*

From Lemma 2.1, we can prove the following theorem by the method similar to the proof in Theorem 3.4 of [5].

THEOREM 2.2. *If $\overline{x} \in X$ is an efficient solution for (P) and if we assume that $\overline{x}$ satisfies a constraint qualification ([7], [8]) for (P_k), $k = 1, \cdots, p$, then there exist $\overline{\lambda} \in \Lambda^+$ and $\overline{y} \in R^m_+$ such that $\nabla\overline{\lambda}^t f(\overline{x}) + \nabla\overline{y}^t g(\overline{x}) = 0$ and $\overline{y}^t g(\overline{x}) = 0$.*

3. Duality Theorems

Now, we establish duality theorems for (P) and (WVD).

THEOREM 3.1. *Suppose that f_i, $i = 1, \cdots, p$ are ρ_i-invex with respect to η and θ, and g_j, $j = 1, \cdots, m$, are σ_j-invex with respect to η and θ. Assume that $\sum_{i=1}^p \lambda_i\rho_i + \sum_{j=1}^m y_j\sigma_j \geq 0$ for all $(u, \lambda, y) \in Y_W$. Then, for all $x \in X$ and for all $(u, \lambda, y) \in Y_W, f(x) \nleq f(u) + y^t g(u)e$.*

Proof. Suppose that there exist $\overline{x} \in X$ and $(\overline{u}, \overline{\lambda}, \overline{y}) \in Y_W$ such that $f(\overline{x}) \leq f(\overline{u}) + \overline{y}^t g(\overline{u})e$. Since $\overline{\lambda} \in \Lambda^+$ and $\overline{y}^t g(\overline{x}) \leq 0$, we have

$$\overline{\lambda}^t f(\overline{x}) - \overline{\lambda}^t f(\overline{u}) + \overline{y}^t g(\overline{x}) - \overline{y}^t g(\overline{u}) < 0.$$

By the $\sum_{i=1}^p \overline{\lambda}_i\rho_i$ -invexity of $\overline{\lambda}^t f(\cdot)$ and the $\sum_{j=1}^m \overline{y}_j\sigma_j$ -invexity of $\overline{y}^t g(\cdot)$, we have

$$\begin{aligned}
&\nabla\overline{\lambda}^t f(\overline{u})\eta(\overline{x},\overline{u}) + \sum_{i=1}^p \overline{\lambda}_i\rho_i\|\theta(\overline{x},\overline{u})\|^2 + \nabla\overline{y}^t g(\overline{u})\eta(\overline{x},\overline{u}) + \sum_{j=1}^m \overline{y}_j\sigma_j\|\theta(\overline{x},\overline{u})\|^2 \\
&= [\nabla\overline{\lambda}^t f(\overline{u}) + \nabla\overline{y}^t g(\overline{u})]\eta(\overline{x},\overline{u}) + (\sum_{i=1}^p \overline{\lambda}_i\rho_i + \sum_{j=1}^m \overline{y}_j\sigma_j)\|\theta(\overline{x},\overline{u})\|^2 < 0.
\end{aligned}$$

Since $\sum_{i=1}^p \overline{\lambda}_i\rho_i + \sum_{j=1}^m \overline{y}_j\sigma_j \geq 0$, $[\nabla\overline{\lambda}^t f(\overline{u}) + \nabla\overline{y}^t g(\overline{u})]\eta(\overline{x},\overline{u}) < 0$. This is a contradiction.

THEOREM 3.2. *Suppose that f_i, $i = 1, \cdots, p$ are ρ_i-invex with respect to η and θ and g_j, $j = 1, \cdots, m$ are σ_j-invex with respect to η and θ. Assume that $\sum_{i=1}^{p} \lambda_i \rho_i + \sum_{j=1}^{m} y_j \sigma_j \geq 0$ for all $(u, \lambda, y) \in Y_W$. Let $\overline{x}$ is an efficient solution for (P) and assume that $\overline{x}$ satisfies a constraint qualification ([7], [8]) for (P_k), $k = 1, \cdots, p$. Then there exist $\overline{\lambda} \in R^p$ and $\overline{y} \in R^m$ such that $(\overline{x}, \overline{\lambda}, \overline{y})$ is an efficient solution for (WVD).*

Proof. By Theorem 2.2, there exist $\overline{\lambda} \in \Lambda^+$ and $\overline{y} \in R_+^m$ such that $(\overline{x}, \overline{\lambda}, \overline{y}) \in Y_W$ and $\overline{y}^t g(\overline{x}) = 0$. By Theorem 3.1, for all $(x, \lambda, y) \in Y_W$, $f(\overline{x}) \not\leq f(x) + y^t g(x)e$. Since $\overline{y}^t g(\overline{x}) = 0$, $f(\overline{x}) + \overline{y}^t g(\overline{x})e \not\leq f(x) + y^t g(x)e$. Hence $(\overline{x}, \overline{\lambda}, \overline{y})$ is an efficient solution for (WVD).

Now, we establish duality theorems for (P) and (MWVD).

THEOREM 3.3. *Suppose that f_i, $i = 1, \cdots, p$ are ρ_i-invex with respect to η and θ and g_j, $j = 1, \cdots, m$ are σ_j-invex with respect to η and θ. Assume that $\sum_{i=1}^{p} \lambda_i \rho_i + \sum_{j=1}^{m} y_j \sigma_j \geq 0$ for all $(u, \lambda, y) \in Y_M$. Then, for all $x \in X$ and for all $(u, \lambda, y) \in Y_M$, $f(x) \not\leq f(u)$.*

Proof. By the method similar to the proof of Theorem 3.1, we can obtain above result.

THEOREM 3.4. *Suppose that f_i, $i = 1, \cdots, p$ are ρ_i-invex with respect to η and θ and g_j, $j = 1, \cdots, m$ are σ_j-invex with respect to η and θ. Assume that $\sum_{i=1}^{p} \lambda_i \rho_i + \sum_{j=1}^{m} y_j \sigma_j \geq 0$ for all $(u, \lambda, y) \in Y_M$. Let $\overline{x}$ is an efficient solution for (P) and assume that $\overline{x}$ satisfies a constraint qualification ([7], [8]) for (P_k), $k = 1, \cdots, p$. Then there exist $\overline{\lambda} \in R^p$ and $\overline{y} \in R^m$ such that $(\overline{x}, \overline{\lambda}, \overline{y})$ is an efficient solution for (MWVD).*

Proof. By Theorem 2.2, there exist $\overline{\lambda} \in \Lambda^+$ and $\overline{y} \in R_+^m$ such that $(\overline{x}, \overline{\lambda}, \overline{y}) \in Y_M$. By Theorem 3.3, for all $(u, \lambda, y) \in Y_M$, $f(\overline{x}) \not\leq f(u)$. Hence $(\overline{x}, \overline{\lambda}, \overline{y})$ is an efficient solution for (MWVD).

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