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A CHARACTERIZATION OF RULED REAL HYPERSURFACES IN $P_n(\mathbb{C})$

YOUNG JIN SUH*

Introduction

Let $P_n(\mathbb{C})$ denote an n-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature $4c$. Real hypersurfaces in $P_n(\mathbb{C})$ have been studied by many differential geometers (See [2], [3], [6], [9] and [11]).

As for a problem concerned with the type number t which is defined by the rank of the second fundamental tensor of real hypersurfaces M in $P_n(\mathbb{C})$, Takagi [9], Yano and Kon [11] showed that there is a point p in M such that $t(p) \geq 2$.

On the other hand, Kimura and Maeda [4] found a non-trivial example of non-homogeneous real hypersurfaces in $P_n(\mathbb{C})$ which is called a ruled real hypersurface. Also it is known that this ruled real hypersurface is not complete and its type number is equal to 2 on the whole M (See Kimura and Maeda [5]). Then it naturally rises to the question that "Is a ruled real hypersurface the only real hypersurface of $P_n(\mathbb{C})$ $(n \geq 3)$ satisfying $t = 2$". The purpose of this paper is to answer this problem affirmatively. Thus as a characterization of a ruled real hypersurface we have the following

THEOREM A. *Let M be a real hypersurface in $P_n(\mathbb{C})$ $(n \geq 3)$ satisfying $t(p) \leq 2$ for any point p in M. Then M is a ruled real hypersurface.*

It is known that a ruled real hypersurface in $P_n(\mathbb{C})$ $(n \geq 3)$ is not complete. Thus, as an application of Theorem A, we also have the following

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THEOREM B. *Let M be a complete real hypersurface in $P_n(\mathbb{C})$ ($n \geq 3$). Then there exists a point p on M such that $t(p) \geq 3$.*

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1. Preliminaries

Let M be a real hypersurface in $P_n(\mathbb{C})$ ($n \geq 2$). Let $\{e_1, \ldots, e_{2n}\}$ be a local field of orthonormal frames in $P_n(\mathbb{C})$ such that, restricted to M, $e_1, \ldots, e_{2n-1}$ are tangent to M. Denote its dual frame field by $\theta_1, \ldots, \theta_{2n}$. We use the following convention on the range of indices unless otherwise stated; $A, B, \ldots, = 1, \ldots, 2n$ and $i, j, \ldots, = 1, \ldots, 2n-1$.

The connection forms θ_{AB} are defined as the 1-forms satisfying

$$d\theta_A = -\sum \theta_{AB} \wedge \theta_B, \quad \theta_{AB} + \theta_{BA} = 0. \tag{1.1}$$

Restrict the forms under consideration to M. Then, we get $\theta_{2n} = 0$ and the forms $\theta_{2n,i}$ can be written as

$$\phi_i \equiv \theta_{2n,i} = \sum h_{ij}\theta_j, \quad h_{ij} = h_{ji}. \tag{1.2}$$

The quadratic form $\sum h_{ij}\theta_i \otimes \theta_j$ is called the *second fundamental form* of M with direction of e_{2n}. The curvature forms Θ_{ij} of M are defined by

$$\begin{aligned} \Theta_{ij} &= d\theta_{ij} + \sum \theta_{ik} \wedge \theta_{kj}, \\ \Theta_{ij} &= \frac{1}{2} \sum R_{ijk\ell}\theta_k \wedge \theta_\ell, \end{aligned} \tag{1.3}$$

where $R_{ijk\ell}$ denotes the component of the Riemannian curvature tensor of M.

We denote by J the complex structure of $P_n(\mathbb{C})$, and put

$$Je_i = \sum J_{ji}e_j + f_i e_{2n}.$$

Then the almost contact structure (J_{ij}, f_k) satisfies

$$\begin{aligned} &\sum J_{ik}J_{kj} = f_i f_j - \delta_{ij}, \quad \sum f_j J_{ji} = 0 \\ &\sum f_i^2 = 1, \quad J_{ij} + J_{ji} = 0. \end{aligned} \tag{1.4}$$

$$(1.5)\qquad \begin{aligned} dJ_{ij} &= \sum(J_{ik}\theta_{kj} - J_{jk}\theta_{ki}) - f_i\phi_j + f_j\phi_i, \\ df_i &= \sum(f_j\theta_{ji} - J_{ji}\phi_j). \end{aligned}$$

The equations of Gauss and Codazzi are given by

$$(1.6)\qquad \begin{aligned} \Theta_{ij} = \phi_i \wedge \phi_j &+ c\theta_i \wedge \theta_j \\ &+ c\sum(J_{ik}J_{j\ell} + J_{ij}J_{k\ell})\theta_k \wedge \theta_\ell, \end{aligned}$$

$$(1.7)\qquad \begin{aligned} d\phi_i = -\sum \phi_j \wedge \theta_{ji} \\ &+ c\sum(f_iJ_{jk} + f_jJ_{ik})\theta_j \wedge \theta_k, \end{aligned}$$

respectively. Then it follows from (1.3) and (1.6) that the components of the Riemannian curvature tensor are given by

$$(1.8)\qquad \begin{aligned} R_{ijk\ell} = c\{\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk} + J_{ik}J_{j\ell} - J_{i\ell}J_{jk} + 2J_{ij}J_{k\ell}\} \\ + h_{ik}h_{j\ell} - h_{i\ell}h_{jk}. \end{aligned}$$

2. Lemmas

Let M be a real hypersurface in $P_n(\mathbb{C})$. We choose an arbitary point p in M, and use the following convention on the range of indices; $a, b, \ldots, = 1, \ldots, t(p)$ and $r, s, \ldots, = t(p)+1, \ldots, 2n-1$. Then we can take a field $\{e_1, \ldots, e_{2n}\}$ of orthonormal frames on a neighborhood of p in such a way that the 1-forms ϕ_i can be written as

$$(2.1)\qquad \begin{aligned} \phi_a &= \sum h_{ba}\theta_b, \quad h_{ab} = h_{ba}, \\ \phi_r &= 0, \end{aligned}$$

at p. We call such a field $\{e_1, \ldots, e_{2n}\}$ to be associated with a point p.

Under this notation we have

LEMMA 2.1. *Assume that $J_{rs}(p) = 0$ at a point p on M. Then $t(p) \geq n-1$. Furthermore, the equality holds if and only if $f_a = 0$ and $J_{ab} = 0$ at p.*

Proof. By (1.4) we have

$$(2.2)\qquad \sum_b J_{ab}^2 + \sum_r J_{ar}^2 + f_a^2 = 1,$$

$$(2.3)\qquad \sum_a J_{ra}^2 + f_r^2 = 1,$$

Summing up (2.2) on a, and (2.3) on r, we have

$$\sum_{a,b} J_{ab}^2 + \sum_{a,r} J_{ar}^2 + \sum_{a} f_a^2 = t(p), \tag{2.4}$$

$$\sum_{a,r} J_{ar}^2 + \sum_{r} f_r^2 = 2n - 1 - t(p). \tag{2.5}$$

Substituting (2.5) into (2.4) and making use of $\sum_a f_a^2 + \sum_r f_r^2 = 1$, we have

$$\sum_{a,b} J_{ab}^2 + 2\sum_{a} f_a^2 = 2(t(p) - (n-1)) \geq 0,$$

and so our assertion follows.

This concludes the proof.

Now we consider a point p where the type number t attains the maximal value, say T. Then there is a neighborhood U of p, on which the function t is constant and the equation (2.1) holds.

Put $\theta_{ar} = \sum A_{arb}\theta_b + \sum B_{ars}\theta_s$. Then, taking the exterior derivative of $\phi_r = 0$ and using (1.7), we have

$$\sum h_{ab}\theta_b \wedge \theta_{ar} - c\sum(f_r J_{ij} + f_i J_{rj})\theta_i \wedge \theta_j = 0,$$

from which we have

$$\sum h_{ab}B_{brs} - cf_a J_{rs} + cf_s J_{ra} - 2cf_r J_{as} = 0, \tag{2.6}$$

$$f_s J_{rt} - f_t J_{rs} + 2f_r J_{st} = 0. \tag{2.7}$$

It is easy to see that (2.7) is reduced to

$$f_r J_{st} = 0. \tag{2.8}$$

Under such a situation we have

LEMMA 2.2. *If $J_{rs} = 0$ on U, then $T \geq n$ on U.*

Proof. If $T < n$, then by Lemma 2.1 we have $T = n - 1$, and $f_a = 0$ on U. For a suitable choice of a field $\{e_r\}$ of orthonormal frames, if necessary, we may set $f_{2n-1} = 1$ and $f_r = 0$ for $r = n, \ldots, 2n-2$. Then from (1.5) we have

$$0 = df_r = -\sum J_{ar}\phi_a.$$

But, since $\operatorname{rank} J = 2n-2$, we have $\det(J_{ar}) \neq 0$ $(a = 1, \ldots, n-1, r = n, \ldots, 2n-2)$. Thus the above equation implies $\phi_a = 0$, which contradicts the fact that $\det(h_{ab}) \neq 0$.

This concludes the proof.

In the remainder of this section we restrict the forms under consideration to the following open set V_T defined by

$$V_T = \{p \in M \mid J_{rs}(p) \neq 0,\ t(p) = T\},$$

where $J_{rs}(p) \neq 0$ means "$J_{rs}(p) \neq 0$ for some $r, s = T+1, \ldots, 2n-1$". First from (2.8) we have $f_r = 0$. Thus we may set $f_1 = 1$, and $f_a = 0$ for $a \geq 2$. Hence from (1.4) we have

$$J_{1a} = 0, \quad J_{1r} = 0. \tag{2.9}$$

Furthermore, $df_r = 0$ gives

$$A_{1ra} = \sum h_{ab}J_{br}, \tag{2.10}$$

$$B_{1rs} = 0. \tag{2.11}$$

The equation (2.6) amounts to

$$\sum h_{ab}B_{brs} = cf_aJ_{rs}. \tag{2.12}$$

LEMMA 2.3. *$\det(h_{ab}) = 0$ $(a, b = 2, \ldots, T)$ on V_T.*

Proof. Here indices a, b run from 2 to T. If $\det(h_{ab}) \neq 0$, then by (2.12) we have $B_{ars} = 0$, which together with (2.11) gives $J_{rs} = 0$. A contradiction to the fact $J_{rs}(p) \neq 0$ on V_T.

This concludes the proof.

3. The proofs of Theorem A and Theorem B

Let M be a real hypersurface of $P_n(\mathbb{C})$ $(n \geq 3)$ with $t(p) \leq 2$ for any point p in M. Let us now construct the following sets which will be used in the later.

$$\begin{aligned} & V = \{p \in M \mid J_{rs}(p) \neq 0\}, \ (r, s, \ldots, = t(p) + 1, \ldots, 2n-1), \\ (3.1) \quad & M_1 = \{p \in M \mid t(p) \leq 1\}, \text{ and} \\ & M_2 = \{p \in M \mid t(p) = 2\}. \end{aligned}$$

Then $M_2 = M - M_1$. Moreover we have $\mathrm{Int}(M_1) = \phi$, because Takagi [9] showed that for any point p in M there exists an open neighborhood U of p in M such that $t(p) \geq 2$, where "Int" means the interior of the given set.

From (3.1) we also construct the following sets

$$(3.2) \qquad V_1 = V \cap M_1 \quad \text{and} \quad V_2 = V \cap M_2.$$

Then V_2 coincides with the open set V_T which is defined in §2 for the case $T = 2$. Since we have assumed $T = 2$, let us restrict the forms under consideration to V_2 unless otherwise stated. Then (2.8) gives $f_r = 0$ for $r = 3, \ldots, 2n-1$, because $J_{rs} \neq 0$. Thus we may set $f_1 = 1$ and $f_a = 0$, $a \geq 2$. From this fact we know that e_1 becomes an almost contact structure vector field.

On the other hand, by Lemma 2.3 we have $h_{22} = 0$ on V_2. From which together with the formula of (2.1) we have

$$(3.3) \qquad A = \left(\begin{array}{cc|c} \alpha & \beta & 0 \\ \beta & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right),$$

that is, $Ae_1 = \alpha e_1 + \beta e_2$ and $Ae_2 = \beta e_1$, where A is the second fundamental tensor of M in $P_n(\mathbb{C})$.

Firstly, we now assert that the holomorphic sectional curvature $H = H(e_i)$ is constant on V_2. Here "the holomorphic section" means the section spanned by $\{e_i, Je_i\}$ for $i \neq 1$. Then the holomorphic sectional curvature is given by

$$H(e_i) = R_{ii}*_{ii}* = \sum_{j,\ell} R_{ijk\ell} J_{ji} J_{\ell i},$$

where e_i* means $Je_i = \sum_j J_{ji}e_j$, $(i \neq 1)$. From which together with (1.4) and (1.8) we have

$$
\begin{aligned}
H(e_i) = c\{1 - f_i^2 + 3(1 - f_i^2)^2\} \\
+ \sum_{j,\ell}(h_{ii}h_{\ell j}J_{\ell i}J_{ji} - h_{ji}h_{\ell i}J_{\ell i}J_{ji}).
\end{aligned} \tag{3.4}
$$

Since $f_i = 0$ and $h_{ii} = 0$ on V_2 for $i = 2, \ldots, 2n-1$, (3.4) reduces to

$$H(e_i) = 4c - \sum_{j,\ell} h_{ji}h_{\ell i}J_{\ell i}J_{ji}.$$

Thus for a case where $i \geq 3$, $H(e_i) = 4c$, because $h_{i\ell} = 0$ for $\ell = 1, \ldots, 2n-1$. For a case where $i = 2$, (2.9) implies that $H(e_2) = 4c$. Thus we have our assertion.

Next we want to show that the holomorphic sectional curvature H is constant on M. The set given in (3.1) becomes

$$V = V \cap M = V \cap (M_1 \cup M_2) = (V \cap M_1) \cup (V \cap M_2) = V_1 \cup V_2.$$

Since $\mathrm{Int}(V_1) = \phi$, from the above formula we have that $H(e_i) = 4c$ on V. Then let us consider an orthogonal complement set of V in M such that $W = M - V$. Thus $J_{rs} = 0$ on $\mathrm{Int}(W)$. From this fact Lemma 2.1 gives $t(p) \geq n - 1 \geq 2$ for any point p in $\mathrm{Int}(W)$. On the other hand, we have assumed $t(p) \leq 2$ for any point p in M. Thus $t(p) = 2$ and $J_{rs}(p) = 0$ on $\mathrm{Int}(W)$. But Lemma 2.2 means that "$J_{rs} = 0$ on $\mathrm{Int}(W)$" implies $T \geq 3$. This makes a contradiction. Consequently, $\mathrm{Int}(W) = \phi$. Thus we conclude that the constancy of the holomorphic sectional curvature H can be extended to M globally.

Now let us recall a theorem which is proved by Kimura [4].

THEOREM C. *Let M be a real hypersurface in $P_n(\mathbb{C})$ $(n \geq 3)$ on which H is constant. Then M is one of the following:*

(a) *an open subset of a geodesic hypersphere $(H > 4c)$,*

(b) *a ruled hypersurface $(H = 4c)$. More precisely, let T_0 be the distribution defined by $T_0(x) = \{X \in T_x(M) \mid X \perp \xi\}$ for $x \in M$, then T_0 is integrable, and its integral manifolds are a totally geodesic $P_{n-1}(\mathbb{C})$.*

(c) *a real hypersurface on which there is a foliation of codimension two such that each leaf of the foliation is contained in some complex hyperplane $P_{n-1}(\mathbb{C})$ as a ruled hypersurface* ($H = 4c$).

It is known that for the case (b) of Theorem C the second fundamental tensor A is given by $A\xi = \alpha\xi + \nu U$, $AU = \nu\xi$ and $AX = 0$ for any X orthogonal to ξ and U, where ξ and U correspond to e_1 and e_2, respectively. Combining this fact with our above assertion we complete the proof of Theorem A. Theorem B immediately follows from Theorem A and the non-completeness of a ruled real hypersurface of the case (b) in Theorem C.

REMARK 1. The above Theorem B is the main result of the paper [7]. In that paper we directly gave the proof of Theorem B by solving a differential equation which is derived from the exterior derivative of (2.1).

REMARK 2. In the paper [8] the present author and Takagi obtained another new rigidity theorem for isometric immersions ι and $\hat{\iota}$ of a real hypersurface M into $P_n(\mathbb{C})$ under the additional condition such that the type number of (M, ι) or $(M, \hat{\iota})$ is not equal to 2 at each point of M. As an application of Theorem B to the homogeneous real hypersurface in $P_n(\mathbb{C})$ $(n \geq 3)$ we also obtained a rigidity Theorem without the above additional condition.

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Department of Mathematics
Andong National University
Andong 760–749, Korea