JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 5, June 1992

A Note on the Lipschitz Classes of Periodic Stochastic Processes of the Second Order

JONG MI CHOO

ABSTRACT. In this paper, T. Kawata's result[2] is generalized, by means of Beurling's norm, in the case of periodic stochastic processes of the second order which belong to Lipschitz class.

1. Introduction

Throughout this paper, $(\Omega, \mathcal{F}, P)$ is the underlying probability space and $X(t,\omega)$, $t \in \mathbf{R}$, is a complex valued stochastic process of the second order, wher ω is an element of Ω, that is

$$E|X(t,\omega)|^2 = \|X(t,\omega)\|^2 < \infty, \quad \text{for every} \quad t.$$

Suppose that $X(t,\omega)$ is measurable on $\mathbf{R} \times \Omega$ and also suppose that

$$\int_a^b \|X(t,\omega)\|^2 \, dt < \infty, \quad \text{for every finite } a < b.$$

We say $X(t,\omega)$ is periodic with period 2π, if

$$\|X(t+2\pi,\omega) - X(t,\omega)\| = 0 \quad \text{for every } t.$$

The class of 2π-periodic processes of the second order will be denoted by L_p^2.

Received by the editors on May 20, 1992.
1980 *Mathematics subject classifications*: Primary 60G12.

Let $\phi(h)$ be a positive nondecreasing function of $h \in (0,1]$. Write

$$\Delta_h^j X(t,\omega) = \sum_{v=0}^{j} (-1)^{j-v} \binom{j}{v} X(t+vh,\omega),$$

where j is a positive integer. The class of $X(t,\omega)$ with the property

$$\sup_{h>0} \int_{-\pi}^{\pi} [\|\Delta_h^j X(t,\omega)\|/\phi(h)]^2 \, dt < \infty$$

is denoted by $\Delta_p^*(\phi)$ and is called the Lipschitz class $\Delta_p^*(\phi)$.

For a stochastic process $X(t,\omega)$ of L_p^2 which belong to $\Delta_p^*(\phi)$, we conside the Fourier series

$$X(t,\omega) \sim \sum_{n=-\infty}^{\infty} C_n(\omega) e^{int},$$

where

$$C_n(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(t,\omega) e^{-int} \, dt.$$

T. Kawata[2] showed that for every positive integer N,

$$(*) \qquad \inf \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\| X(t,\omega) - \sum_{n=-N}^{N} a_n(\omega) e^{int} \right\|^2 dt$$

is attained by $a_n(\omega) = C_n(\omega)$, a.s., where $C_n(\omega)$ is the Fourier coefficient of $X(t,\omega)$ as before and the inf is taken over all random variables $a_n(\omega) \in L^2(\Omega)$. Writing the quantity $(*)$ by $E_N^{(2)}$, we have

$$E_N^{(2)} = \left[\sum_{|n| \geq N} \|C_n(\omega)\|^2 \right]^{\frac{1}{2}}$$

and we call it as the best approximation by trigonometric polynomials in L_p^2.

T. Kawata[2] proved the following theorem.

THEOREM. *If* $\sum_{n=1}^{\infty} n^{-\frac{1}{2}} E_n^{(2)} < \infty$ *then*

$$\sum_{n=-\infty}^{\infty} |C_n(\omega)| < \infty, \qquad a.s.$$

The purpose of this paper is to extend the above result using the method of M. Kinukawa[4] and R. G. Mamedov and G. I. Osamanov[5].

For the convenience of the subsequent study we shall mention the definition of Beurling's norm.

Let $0 < p \leq a < \infty$.

$$W = \left\{(\omega_n) : \omega_n > 0, \omega_n = \omega_{-n}, \omega_{|n|} \downarrow, \|\omega_n\|_1 = \sum_{n=-\infty}^{\infty} \omega_n < \infty\right\},$$

$$_a\|c_n\|_{p,\omega} = \left\{\sum_{n=-\infty}^{\infty} |c_n|^a (\omega_n)^{1-\frac{a}{p}}\right\}^{\frac{1}{a}},$$

$$_a\|c_n\|_p = \inf_{(\omega_n)\in W} [\|\omega_n\|_1^{\frac{1}{p}-\frac{1}{a}} {}_a\|c_n\|_{p,\omega}].$$

$$_a\ell_p = \{(c_n) :_a \|c_n\|_p < \infty\}.$$

$_a\|\cdot\|_p$ is called the Beurling's norm. By Hölder's inequality, we see that $\|c_n\|_p \leq_a \|c_n\|_p$. In particular, $\|c_n\|_p =_p \|c_n\|_p$. The above inequality holds for the case $0 < p < \infty = a$. Because, we read

$$_\infty\|c_n\|_p = \inf_{(\omega_n)} [\|\omega_n\|^{\frac{1}{p}} \sup_n (|c_n|\omega_n^{-\frac{1}{p}})]$$

and we have

$$\sum |c_n|^p = \sum (|c_n|\omega_n^{-\frac{1}{p}})^p \omega_n \leq \|\omega_n\|_1 \left[\sup_n (|c_n|\omega_n^{-\frac{1}{p}})\right]^p.$$

2. Theorem

We shall use the following notations:

$$ {}_2A_{1,j}(X) = \int_0^1 h^{-\frac{1}{2}} \left[\int_{-\pi}^{\pi} \|\Delta_h^j X(t,\omega)\|^2 \, dt \right]^{\frac{1}{2}} dh $$

$$ {}_2B_1(X) = \sum_{n=1}^{\infty} n^{-\frac{1}{2}} \left(\sum_{|k| \geq n+1} \|C_k(\omega)\|^2 \right)^{\frac{1}{2}} $$

$$ {}_2C_{1,j}(X) = \sum_{n=1}^{\infty} n^{-j+\frac{3}{2}} \left(\sum_{|k| \leq n} k^j \|C_k(\omega)\|^2 \right)^{\frac{1}{2}} $$

$$ {}_2B_1^*(X) = \sum_{m=0}^{\infty} 2^{-\frac{m}{2}} \left(\sum_{k=2^m}^{\infty} \|C_k(\omega)\|^2 \right)^{\frac{1}{2}} $$

$$ {}_2C_{1,j}^*(X) = \sum_{m=0}^{\infty} 2^{-m(j+\frac{1}{2})} \left(\sum_{k=1}^{2^m} k^{2j} \|C_k(\omega)\|^2 \right)^{\frac{1}{2}}. $$

THEOREM. *The following two relations are equivalent:*

$$ \sum_{n=1}^{\infty} n^{-\frac{1}{2}} E_n^{(2)} < \infty $$

and

$$ {}_2\| \, \|C_n(\omega)\| \, \|_1 < \infty. $$

For the proof of Theorem, we now give two lemmas. (K is a constant which may be different in each occurence.)

LEMMA 1. *The conditions ${}_2B_1(X) < \infty$ and ${}_2C_{1,j}(X) < \infty$ are, respectively, equivalent to the relations ${}_2B_1^*(X) < \infty$ and ${}_2C_{1,j}^*(X) < \infty$.*

LEMMA 2. *The following two relations are equivalent:*

$$ {}_2B_1^*(X) < \infty \quad \text{and} \quad {}_2C_{1,j}^*(X) < \infty. $$

PROOF. Suppose that ${}_2B_1^*(X) < \infty$. Then

$$ \begin{aligned} {}_2C_{1,j}^*(X) &= \sum_{m=0}^{\infty} 2^{-m(j+\frac{1}{2})} \left(\sum_{k=1}^{2^m} k^{2j} \|C_k(\omega)\|^2 \right)^{\frac{1}{2}} \\ &\leq \sum_{m=0}^{\infty} 2^{-m(j+\frac{1}{2})} \left\{ \sum_{v=0}^{m} \left(\sum_{k=2^v}^{2^{v+1}} k^{2j} \|C_k(\omega)\|^2 \right) \right\}^{\frac{1}{2}} \\ &\leq K \sum_{m=0}^{\infty} 2^{-m(j+\frac{1}{2})} \sum_{v=0}^{m} 2^{vj} \left(\sum_{k=2^v}^{2^{v+1}} \|C_k(\omega)\|^2 \right)^{\frac{1}{2}} \\ &= K \sum_{v=0}^{\infty} 2^{vj} \left(\sum_{k=2^v}^{2^{v+1}} \|C_k(\omega)\|^2 \right)^{\frac{1}{2}} \sum_{m=v}^{\infty} 2^{-m(j+\frac{1}{2})} \\ &\leq K \sum_{v=0}^{\infty} 2^{-\frac{v}{2}} \left(\sum_{k=2^v}^{2^{v+1}} \|C_k(\omega)\|^2 \right)^{\frac{1}{2}} \\ &\leq K \sum_{v=0}^{\infty} 2^{-\frac{v}{2}} \left(\sum_{k=2^v}^{\infty} \|C_k(\omega)\|^2 \right)^{\frac{1}{2}} \\ &= K {}_2B_1^*(X). \end{aligned} $$

Let us prove the converse:

$$ \begin{aligned} {}_2B_1^*(X) &= \sum_{m=0}^{\infty} 2^{-\frac{m}{2}} \left(\sum_{k=2^m}^{\infty} \|C_k(\omega)\|^2 \right)^{\frac{1}{2}} \\ &= \sum_{m=0}^{\infty} 2^{-\frac{m}{2}} \left(\sum_{v=m}^{\infty} \sum_{k=2^v}^{2^{v+1}} \|C_k(\omega)\|^2 \right)^{\frac{1}{2}} \\ &\leq \sum_{m=0}^{\infty} 2^{-\frac{m}{2}} \sum_{v=m}^{\infty} 2^{-jv} \left(\sum_{k=2^v}^{2^{v+1}} k^{2j} \|C_k(\omega)\|^2 \right)^{\frac{1}{2}} \\ &= \sum_{v=0}^{\infty} 2^{-jv} \left(\sum_{k=2^v}^{2^{v+1}} k^{2j} \|C_k(\omega)\|^2 \right)^{\frac{1}{2}} \sum_{m=0}^{v} 2^{-\frac{m}{2}} \end{aligned} $$

$$\leq K \sum_{v=0}^{\infty} 2^{-v(j+\frac{1}{2})} \left(\sum_{k=1}^{2^{v+1}} k^{2j} \|C_k(\omega)\|^2 \right)^{\frac{1}{2}}$$
$$= K_2 C^*_{1,j}(X).$$

Now we will prove our main theorem.

PROOF OF THEOREM. The Fourier coefficient of $\Delta_h^j x(t,\omega)$ is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \Delta_h^j X(t,\omega) e^{-int}\, dt$$
$$= \sum_{k=0}^{j} (-1)^k \binom{j}{k} \frac{1}{2\pi} \int_{-\pi}^{\pi} X(t+kh,\omega) e^{-int}\, dt$$
$$= \sum_{k=0}^{j} (-1)^k \binom{j}{k} \frac{1}{2\pi} \int_{-\pi}^{\pi} X(t,\omega) e^{-int} dt e^{inkh}$$
$$= C_n(\omega)(1-e^{inh})^j.$$

By the Parseval equation, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |\Delta_h^j X(t,\omega)|^2\, dt = \sum_{n=-\infty}^{\infty} |C_n(\omega)|\, |e^{inh}-1|^{2j}.$$

Taking expectations of both sides, we have

$$E\left[\sum_{n=-\infty}^{\infty} |C_n(\omega)|\, |e^{inh}-1|^{2j}\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \|\Delta_h^j X(t,\omega)\|^2\, dt.$$

And

$$E\left[\sum_{n=-\infty}^{\infty} |C_n(\omega)|\, |e^{inh}-1|^{2j}\right] = \sum_{n=-\infty}^{\infty} \|C_n(\omega)\|^2\, |e^{inh}-1|^{2j}$$
$$= 2^{2j} \sum_{n=-\infty}^{\infty} \|C_n(\omega)\|^2 \left[\sin\left(\frac{nh}{2}\right)\right]^{2j}.$$

Now

$$
\begin{aligned}
{}_2A_{1,j}(X) &= \int_0^1 h^{-\frac{1}{2}} \left[\int_{-\pi}^{\pi} \|\Delta_h^j X(t,\omega)\|^2\, dt\right]^{\frac{1}{2}} dh \\
&\leq K \int_0^1 h^{-\frac{1}{2}} \left(\sum_{|k|=1}^{n} \|C_k(\omega)\|^2 \left|\sin\frac{kh}{2}\right|^{2j}\right)^{\frac{1}{2}} dh \\
&+ K \int_0^1 h^{-\frac{1}{2}} \left(\sum_{|k|>n} \|C_k(\omega)\|^2 \left|\sin\frac{kh}{2}\right|^{2j}\right)^{\frac{1}{2}} dh \\
&\leq K \sum_{n=1}^{\infty} n^3 \int_{\frac{1}{n+1}}^{\frac{1}{n}} h^{j-\frac{1}{2}}\, dh \left(\sum_{|k|=1}^{n} k^{2j}\|C_k(\omega)\|^2\right)^{\frac{1}{2}} \\
&+ K \sum_{n=1}^{\infty} n^3 \int_{\frac{1}{n+1}}^{\frac{1}{n}} h^{-\frac{1}{2}}\, dh \left(\sum_{|k|>n} \|C_k(\omega)\|^2\right)^{\frac{1}{2}} \\
&\leq K \sum_{n=1}^{\infty} n^{-j+\frac{3}{2}} \left(\sum_{|k|=1}^{n} k^{2j}\|C_k(\omega)\|^2\right)^{\frac{1}{2}} \\
&+ K \sum_{n=1}^{\infty} n^{\frac{1}{2}} \left(\sum_{|k|>n} \|C_k(\omega)\|^2\right)^{\frac{1}{2}} \\
&= K_2 C_{1,j}(X) + K_2 B_1(X).
\end{aligned}
$$

By Lemma 1, Lemma 2 and Theorem 3[4], the proof is complete.

REFERENCES

[1] T. Kawata, *Lipschitz classes of periodic stochastic processes*, Keio Math. Sem. Rep. No.3 (1978), 33–43.

[2] ———, *Absolute convergence of Fourier series of periodic stochastic processes and its applications*, Tohoku Math. J. **35** (1983), 459–474.

[3] ———, *Lipschitz Classes and Fourier Series of Stochastic Processes*, Tokyo J. Math. Vol.11, No.2 (1988), 269–280.

[4] M. Kinukawa, *Some generalizations of contraction theorems for Fourier series*, Pacific J. Math. **109** (1983), 121–134.

[5] R. G. Mamedov and G. I. Osamanov, *Some properties of functions expressible by their Fourier coefficients and transforms*, Izvestiya VUZ. Mathematika **20** (1976), 65–78.

Department of Mathematics
Mokwon University
Taejon, 301-729, Korea