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GENERALIZED FRACTIONS AND REGULAR SEQUENCES

YEONG MOO SONG

INTRODUCTION

In [8], Sharp and Zakeri introduced a concept of module of generalized fractions in commutative algebra. The concept of module of generalized fractions interacts astonishingly well with the theory of regular sequences in commutative algebra, and in this paper we develop some of the connections.

Let A be a commutative ring with identity and M an A-module. One of the reasons why generalized fractions can play a substantial role in the theory of regular sequences is the injectivity of the determinantal homomorphism, which is proved complete generality by Gibson in [2, 3.4]. It was also proved for the particular case where A is Noetherian ring and M is finitely generated, by O'Carroll in [3, 3.7], and in [1, p.690] for the case where M is the ring A itself.

In the first section of this paper we give the brief résumé of the theory of Sharp and Zakeri, and also review the important concept of the saturation of a triangular subset. In the second section we show that the determinantal homomorphism induces an isomorphism. finally, we use generalized fractions to compare length of maximal M-sequences contained in a specific ideal.

1. PRELIMINARIES

Throughout this paper, A is a commutative ring with identity and M is an A-module.The positive integers are denoted by $\mathbb{N}$ with n as a typical element, while $Ln(A)$ denotes the set of $n \times n$ lower triangular matrices over A. As usual, $|H|$ denotes the determinant of a matrix $H \in Ln(A)$, while ^{T} denotes matrix transpose.

A non-empty subset U of A^n is called *triangular* if

(i) given $(u_1, \cdots, u_n) \in U, ({u_1}^{\alpha_1}, \cdots, {u_n}^{\alpha_n}) \in U$ for all $\alpha_i \in \mathbb{N}, 1 \leq i \leq n$;

(ii) given $(u_1, \cdots, u_n)$ and $(v_1, \cdots, v_n)$ in U, there exists $(w_1, \cdots, w_n) \in U$ and $H, K \in Ln(A)$ such that

$$H[u_1 \cdots u_n]^T = [w_1 \cdots w_n]^T = K[v_1 \cdots v_n]^T.$$

Whenever we can do so without ambiguity we shall denote $(u_1, \cdots, u_n)$ by u, and $[u_1 \cdots u_n]^T$ by u^T, and we shall use obvious extensions of this notations.

Given such a triangular subset U of A^n, we can form the module of generalized fractions $U^{-n}M = \{m/u | m \in M, u \in U\}$, where m/u denotes the equivalence class of the pair $(m, u) \in M \times U$ under the following equivalence relation $\sim$ on $M \times U$:

$(c, u) \sim (d, v)$ precisely when there exist $w \in U$ and $P, Q \in Ln(A)$ such that $Pu^T = w^T = Qv^T$, with $|P|c - |Q|d \in \sum_1^{n-1} w_i M$.

* Associate Professor, Department of Mathematics Education, Sunchun National University.

Now $U^{-n}M$ is an A-module under the operations

$$a/u + b/v = (|H|a + |K|b)/w,$$
$$r \cdot (a/u) = ra/u,$$

for $r \in A, a, b \in M, u, v \in U$, and any choice of $H, K \in Ln(A)$ and $w \in U$ such that $Hu^T = w^T = Kv^T$. Furthermore we shall need the following basic property which give a hint about the useful role modules of generalized fractions can play in the theory of regular sequences.

PROPOSITION 1.1. *([7,2.2] & [8,3.3]) Assume that the triangular subset U of A^n consists entirely of poor M-sequences. Let $m \in M$ and $u = (u_1, \cdots, u_n) \in U$. Then*

$$m/u = 0 \text{ in } U^{-n}M \text{ if and only if } m \in \sum_{1}^{n-1} u_i M.$$

In [4,§2], Riley introduces the idea of a saturated triangular subset and gives interesting properties of these. Given a triangular subset U the *saturation* of U is

$$\tilde{U} = \{v = (v_1, \cdots, v_n) \in A^n \mid \text{there exist } H \in Ln(A) \text{ and } u \in U \text{ such that } Hv^T = u^T\}.$$

The triangular subset U is *saturated* if $\tilde{U} = U$. One of the properties of saturated triangular subsets is stated in the following proposition.

PROPOSITION 1.2. *([4,2.9]) Let V and U be triangular subset of A^n with $V \subseteq U \subseteq \tilde{V}$. Then the natural A-homomorphism*

$$\sigma : V^{-n}M \longrightarrow U^{-n}M$$

given by $\sigma(m/u) = m/u$ for all $m \in M, u \in U$, is an isomorphism.

2. Generalized fractions and determinantal homomorphisms.

NOTATION 2.1. *Suppose that $x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n) \in A^n$ and $H \in Ln(A)$ are such that $Hx^T = y^T$. By [8,2.2],*

$$|H|(\sum_{1}^{n} x_i M) \subseteq \sum_{1}^{n} y_i M.$$

Hence there is induced an A-homomorphism

$$\alpha_H : M/\sum_{1}^{n} x_i M \longrightarrow M/\sum_{1}^{n} y_i M$$

*for which $\alpha_H(m + \sum_1^n x_i M) = |H|m + \sum_1^n y_i M$ for each $m \in M$. The homomorphism α_H will be called the **determinantal homomorphism** (induced by H).*

PROPOSITION 2.2. *([3,3.2]) Suppose that $x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n) \in A^n$ are such that*

(i) $y^T = Hx^T$ for some $H \in Ln(A)$, and

(ii) y is a poor M-sequence.

Then the homomorphism

$$\alpha_H : M/\sum_1^n x_i M \longrightarrow M/\sum_1^n y_i M$$

of 2.1 is a monomorphism and x is a poor M-sequence.

COROLLARY 2.3. *Let $x = (x_1, \cdots, x_n)$ be a poor M-sequence, and let*

$$U_x = \{(x_1^{r_1}, \cdots, x_n^{r_n}) | r_1, \cdots, r_n \in \mathbb{N}\}$$

be the triangular subset of A^n. Then its saturation $\tilde{U}_x$ consists entirely of poor M-sequences.

DEFINITION AND REMARKS 2.4. Let A be Noetherian and M be a non-zero A-module. Let $\mathfrak{a}$ be an ideal of A. Let $x_1, \cdots, x_n$ form an M-sequence of elements of $\mathfrak{a}$. (We say "*M-sequence (contained) in $\mathfrak{a}$*".)

(i) We say that $x_1, \cdots, x_n$ form a *maximal M-sequence (contained) in* $\mathfrak{a}$ if there does not exist $x_{n+1} \in \mathfrak{a}$ such that $x_1, \cdots, x_n, x_{n+1}$ form an M-sequence. If $M/\mathfrak{a}M \neq 0$, then $x_1, \cdots, x_n$ form a maximal M-sequence contained in $\mathfrak{a}$ if and only if $\mathfrak{a} \subseteq \mathfrak{Z}(M/\sum_1^n x_i M)$ (We use $\mathfrak{Z}(M)$ to denote the set of zero divisors on M.)

(ii) There exist M-sequences contained in $\mathfrak{a}$, for the empty M-sequence is one such. Also, by [6,3.3], every M-sequence contained in $\mathfrak{a}$ can be extended to a maximal such.

NOTATION 2.5. *Let U be a triangular subset of A^n and let $\mathfrak{a}$ be an ideal of A. We shall denote $\{\delta \in U^{-n}M | \mathfrak{a}\delta = 0\}$ by $Ann(\mathfrak{a}, U^{-n}M)$, read as "the annihilator of $\mathfrak{a}$ in $U^{-n}M$".*

PROPOSITION 2.6. *Let $\mathfrak{a}$ be an ideal of A and let U be a triangular subset of A^n which consists entirely of poor M-sequences. Suppose that $m \in M, x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n) \in U$ and $H = (h_{ij}) \in Ln(A)$ are such that*

$$\frac{m}{(y,1)} \in Ann(\mathfrak{a}, (U \times \{1\})^{-n-1}M),$$

that $Hx^T = y^T$ and that $x_1, \cdots, x_n \in \mathfrak{a}$. Then

$$\frac{m}{(y,1)} = \frac{m'}{(x,1)}$$

for some $m' \in M$.

Proof. By 1.2 and 2.2, we can suppose that U is saturated. Then $(x_1, \cdots, x_i, y_{i+1}, \cdots, y_n) \in U$ (for each $i = 1, \cdots, n$) by virtue of the fact that $H_1[x_1 \cdots x_i y_{i+1} \cdots y_n]^T = [y_1 \cdots y_n]^T$, where

$$H_1 = \begin{bmatrix} h_{11} & & & & & \\ \vdots & \ddots & & & 0 & \\ h_{i1} & \dots & h_{ii} & & & \\ & & & 1 & & \\ & 0 & & & \ddots & \\ & & & & & 1 \end{bmatrix}$$

Now, the proposition is proved by a straightforward inductive process, and the details are left to the reader. □

THEOREM 2.7. *Let $\mathfrak{a}$ be an ideal of A and let $x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n) \in A^n$ be such that there exists $H \in Ln(A)$ with $Hx^T = y^T$. Suppose that y is a poor M-sequence and $x_1, \cdots, x_n \in \mathfrak{a}$ (so that, by 2.2, both x and y are poor M-sequences contained in $\mathfrak{a}$). Then the determinantal homomorphism α_H of 2.2 induces an isomorphism*

$$\alpha_H^* = Ann(\mathfrak{a}, \alpha_H) : (xM : \mathfrak{a})/xM \longrightarrow (yM : \mathfrak{a})/yM.$$

Proof. Of course $(xM : \mathfrak{a})/xM = Ann(\mathfrak{a}, M/xM)$, etc., and $\alpha_H^*(t + xM) = |H|t + yM$ for all $t \in (xM : \mathfrak{a})$. It is immediate from 2.2 that α_H^* is a monomorphism, and we show now that it is surjective.

Let $s \in (yM : \mathfrak{a})$. We work with the triangular subset

$$V = \tilde{U}_y \times \{1\}$$

of A^{n+1} and we note that $(y, 1) \in V$ and $(x, 1) \in V$ also since $Hx^T = y^T$. The generalized fraction $\frac{s}{(y,1)} \in V^{-n-1}M$ is annihilated by $\mathfrak{a}$. because $s \in (yM : \mathfrak{a})$. By 2.3, V consists entirely of poor M-sequences; it therefore follows from 2.6 that there exists $m \in M$ such that, in $V^{-n-1}M$,

$$\frac{s}{(y,1)} = \frac{m}{(x,1)}.$$

Since this generalized fraction is annihilated by $\mathfrak{a}$, it follows from 1.1 that $m \in (xM : \mathfrak{a})$. Also, because $Hx^T = y^T$, we have

$$\frac{s}{(y,1)} = \frac{m}{(x,1)} = \frac{|H|m}{(y,1)};$$

another use of 1.1 therefore shows that $s - |H|m \in yM$; this shows that α_H^* is surjective. □

REMARKS 2.8. Let the situation be as in 2.7,so that $x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n)$ are poor M-sequences contained in the ideal $\mathfrak{a}$ of A such that $Hx^T = y^T$ for some $H \in Ln(A)$.

(i) The A-homomorphism

$$\phi : (xM : \mathfrak{a}) \longrightarrow (U_x \times \{1\})^{-n-1}M$$

defined by $\phi(t) = \frac{t}{(x,1)}$ for all $t \in (xM : \mathfrak{a})$ has kernel xM, by 1.1; moreover the same result shows that $Im\phi \subseteq Ann(\mathfrak{a}, (U_x \times \{1\})^{-n-1}M)$. We now show that $Im\phi = Ann(\mathfrak{a}, (U_x \times \{1\})^{-n-1}M)$ by means of 2.6: it is easy to see that $U_x \times \{1\}$ consists entirely of poor M-sequences, and an arbitrary element ξ of $Ann(\mathfrak{a}, (U_x \times \{1\})^{-n-1}M)$ has the form

$$\frac{t}{(x_1^{r_1}, \cdots, x_n^{r_n}, 1)}$$

for some $t \in M$ and $r_1, \cdots, r_n \in \mathbb{N}$. Since $D[x_1 \cdots x_n]^T = [x_1^{r_1} \cdots x_n^{r_n}]^T$, where D is the diagonal matrix $diag[x_1^{r_1-1} \cdots x_n^{r_n-1}]$, it follows from 2.6 and 1.1 that $\xi \in Im\phi$. Thus there exists an A-isomorphism

$$\psi_x : (xM : \mathfrak{a})/xM \longrightarrow Ann(\mathfrak{a}, (U_x \times \{1\})^{-n-1}M)$$

which is such that $\psi_x(t + xM) = \frac{t}{(x,1)}$ for all $t \in (xM : \mathfrak{a})$.

(ii) Note that $x \in \tilde{U}_y$, so that

$$U_x \times \{1\} \subseteq \tilde{U}_y \times \{1\}.$$

In fact, if we let V be any triangular subset of A^n such that $x \in V \subseteq \tilde{U}_y$, so that

$$U_x \times \{1\} \subseteq V \times \{1\} \subseteq \tilde{U}_y \times \{1\},$$

then it is easy to check that the diagram

$$\begin{array}{ccc}
(xM : \mathfrak{a})/xM & \xrightarrow[\simeq]{\psi_x} & Ann((\mathfrak{a}, (U_x \times \{1\})^{-n-1}M)) \\
 & & \downarrow \chi_1 \\
 & & Ann((\mathfrak{a}, (V \times \{1\})^{-n-1}M)) \\
\alpha_H^* \downarrow \simeq & & \downarrow \chi_2 \\
 & & Ann((\mathfrak{a}, (\tilde{U}_y \times \{1\})^{-n-1}M)) \\
 & & \simeq \uparrow \chi_3 \\
(yM : \mathfrak{a})/yM & \xrightarrow[\simeq]{\psi_y} & Ann((\mathfrak{a}, (U_y \times \{1\})^{-n-1}M))
\end{array}$$

in which α_H^* is the isomorphism of 2.7 and χ_1, χ_2, χ_3 are the natural monomorphisms, commutes. By 1.2, χ_3 is an isomorphism. It therefore follows that χ_1 and χ_2 are also isomorphisms.

3. Length of maximal M-sequence

In this section we are going to introduce some Noetherian hypotheses under which we can give necessary and sufficient conditions for the vanishing of the module $(xM : \mathfrak{a})/xM$ of 2.8. Probably the most important situation for the developement of the theory of grade is that where A is Noetherian and M is finitely generated. However, at least part of the theory can be developed in more general situations without much additional effort, and in order to do this we make the following definition.

DEFINITION 3.1. Let A be Noetherian. We say that an A-module M is *avoidant* if, whenever $\mathfrak{a}$ is an ideal of A and $x = (x_1, \cdots, x_n)$ is a poor M-sequence contained in $\mathfrak{a}$ such that $\mathfrak{a} \subseteq \mathfrak{Z}(M/xM)$, then $\mathfrak{a} \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in Ass(M/xM)$.

EXAMPLES 3.2. Let A be Noetherian

(i) The most important example of an avoidant A-modules is a finitely generated A-module M: then, for every poor M-sequence $x = (x_1, \cdots, x_n)$, the set $Ass(M/xM)$ is finite; since

$$\mathfrak{Z}(M/xM) = \bigcup_{\mathfrak{p} \in Ass(M/xM)} \mathfrak{p},$$

it follows from ordinary "prime avoidance" that M is avoidant.

(ii) Any countably generated module over a complete local ring A is an avoidant A-module: See [6,3.2 and 2.2].

For the rest of this section, assume that A is Noetherian and M is an avoidant A-module.

LEMMA 3.3. *Let $x = (x_1, \cdots, x_n)$ be a poor M-sequence composed of elements of the ideal $\mathfrak{a}$ of A. Then $(xM : \mathfrak{a}) = xM$ if and only if there exists $x_{n+1} \in \mathfrak{a}$ such that $(x, x_{n+1}) = (x_1, \cdots, x_n, x_{n+1})$ is a poor M-sequence.*

Proof. ($\Leftarrow$) Suppose that there exists $x_{n+1} \in \mathfrak{a}$ such that (x, x_{n+1}) is a poor M-sequence. Let $m \in (xM : \mathfrak{a})$. Then $x_{n+1}m \in xM$, so that $m \in xM$ since (x, x_{n+1}) is a poor M-sequence.

($\Rightarrow$) Assume that $(xM : \mathfrak{a}) = xM$. Suppose that there does not exist $x_{n+1} \in \mathfrak{a}$ such that (x, x_{n+1}) is a poor M-sequence. This means that $\mathfrak{a} \subseteq \mathfrak{Z}(M/xM)$, so that, since M is avoidant, $\mathfrak{a} \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in Ass(M/xM)$. Now there exists an element of M/xM whose annihilator is exactly $\mathfrak{p}$, and so $Ann(\mathfrak{a}, M/xM) \neq 0$. Thus $(xM : \mathfrak{a}) \neq xM$ and with this contradiction the proof is complete. □

The next proposition can be proved easily by induction, and is of assitance in the proof of Theorem 3.5.

PROPOSITION 3.4. *The set of all poor M-sequence of length n is a triangular subset of A^n.*

Observe that if $\mathfrak{a}$ is an ideal of A such that $M \neq \mathfrak{a}M$, then any poor M-sequence contained in $\mathfrak{a}$ is automatically an M-sequence in $\mathfrak{a}$.

THEOREM 3.5. *Let $\mathfrak{a}$ be an ideal of A such that $M \neq \mathfrak{a}M$. Then any two maximal M-sequence contained in $\mathfrak{a}$ have the same length.*

Proof. In view of 2.4(ii), it is enough to show that if $(x_1, \cdots, x_{n-1}, x_n)$ and $s' = (s_1, \cdots, s_{n-1})$ are M-sequences contained in $\mathfrak{a}$, then there exists $s_n \in \mathfrak{a}$ such that (s', s_n) is an M-sequence. This we do.

Note that $(s', 1)$ is a poor M-sequence. Therefore, by 3.4, there exist a poor M-sequence $(y_1, \cdots, y_{n-1}, y_n)$ and $H, K \in Ln(A)$ such that

$$H[x_1 \cdots x_{n-1} x_n]^T = [y_1 \cdots y_{n-1} y_n]^T = K[s_1 \cdots s_{n-1} 1]^T.$$

Since $y_j \in \sum_1^j Ax_i \subseteq \mathfrak{a}$ for each $j = 1, \cdots, n$, it is in fact the case that $(y_1, \cdots, y_{n-1}, y_n)$ is an M-sequence in $\mathfrak{a}$. Write $y' = (y_1, \cdots, y_{n-1})$. By 2.7,

$$(s'M : \mathfrak{a})/s'M \simeq (y'M : \mathfrak{a})/y'M,$$

which is zero by 3.3, since (y', y_n) is an M-sequence in $\mathfrak{a}$. Hence $(s'M : \mathfrak{a})/s'M = 0$ and so, by 3.3 again, there exists $s_n \in \mathfrak{a}$ such that (s', s_n) is a poor M-sequence. Since (s', s_n) must, in fact, be an M-sequence, the proof is complete. □

Let $\mathfrak{a}$ be an ideal of A such that $M \neq \mathfrak{a}M$. Then the common length (see 3.5) of all maximal M-sequence in $\mathfrak{a}$ is called the *M-grade of* $\mathfrak{a}$ and denoted by $grade_M \mathfrak{a}$. In this situation, we have now established the following important facts : there exist M-sequence in $\mathfrak{a}$, and every M-sequence contained in $\mathfrak{a}$ can be extended to a maximal such, which will have $grade_M \mathfrak{a}$ terms.

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DEPARTMENT OF MATHEMATICS EDUCATION, SUNCHUN NATIONAL UNIVERSITY SUNCHUN 540-742, KOREA