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DEGREES OF IRREDUCIBLE CHARACTERS AND NORMAL π–COMPLEMENT

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1. INTRODUCTION

Let G be a finite group and let c.b.(G) be the set of degrees of irreducible complex characters of G.

In[1], Ya.G.Berkovich obtained a theorem :

Theorem. *(Berkovich) Let B be the intersection of the kernels of all nonlinear irreducible complex characters of G with p'-degree. Then $B \cap G' \cap P \subseteq P'$, where P is a Sylow p-subgroup of G. Also B has a normal p-complement.*

Using the above theorem, Berkovich proved the following well known theorem in different way.

Theorem. *(Thompson) Suppose that p devides $\chi(1)$ for every nonlinear irreducible character χ of G. Then G has a normal p-complement.*

In this paper, we generalize a part of the above Berkovich's Theorem and we obtain a result as follows :

Theorem 1.1. *Let π be a set of primes. Let B be the intersection of the kernels of all nonlinear irreducible complex characters of G with π'-degree. Let S be a π-subgroup of G. Then $B \cap G' \cap S \subseteq S'$.*

Theorem 1.2. *Let G be a finite solvable group and let $\pi = \pi(c.d.(G))$. Then G has a normal π-complement and $G' \cap S = S'$, where S is a S_π-subgroup of G.*

For notations and terminologies not described, one confer [2] and [3]. In particular, we denote by $Irr(G)$ the set of all irreducible characters of G. All characters are considered in this paper are complex characters and all groups are assumed to be finite groups.

2. PRELIMINARIES

Let G be a group and let $GL(n, C)$ be the general linear group of degree n over the complex number field C. A representation of G is a homomorphism Φ from G into $GL(n, C)$. The character χ of G afforded by Φ is the function given by $\chi(g) = tr\Phi(g)$ for $g \in G$.

Let χ be a character of G. Then $\chi(1)$ is called the degree of χ and the set $\ker\chi = \{g \in G | \chi(g) = \chi(1)\}$ is called the kernel of χ.

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There is a natural way to build modules from representations and representations from modules (cf.[2],[3]). A representation is said to be irreducible (reducible) according as the module built from the representation is irreducible (reducible).

Let χ be a character afforded by Φ. We call χ irreducible (reducible) according as Φ is irreducible (reducible). A character of degree 1 is called a linear character. A linear character is irreducible and irreducible character is linear if and only if its kernel contains the commutator subgroup.

For convenience, we describe the following two definitions.

Definition 2.1. Let χ and ψ be class functions on a group G. The inner product of χ and ψ is defined by

$$[\chi, \psi] = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)}$$

, where $\overline{\psi(g)}$ is the complex conjugate of $\psi(g)$.

Definition 2.2. Let H be a subgroup of a group G. Let χ be a character of H. The induced charater χ^G of χ is defined by

$$\chi^G(g) = \frac{1}{|H|} \sum_{x \in G} \mathring{\chi}(xgx^{-1})$$

,where $\mathring{\chi}$ is defined by $\mathring{\chi}(h) = \chi(h)$ if $h \in H$ and $\mathring{\chi}(y) = 0$ if $y \notin H$.

We describe some lemmas without proof, which is necessary in section 3.

Lemma 2.3. *Let χ be a character of G with $\chi = \sum n_i \chi_i$ for $\chi_i \in Irr(G)$. Then* $\ker \chi = \cap\{\ker \chi_i | n_i > 0\}$.

proof. [3, Lemma 2.21].

Lemma 2.4. *Let G be a group with commutator subgroup G'. Then*

$$G' = \cap\{\ker \lambda | \lambda \in Irr(G), \lambda(1) = 1\}.$$

proof. [3, corollary 2.23].

Lemma 2.5. *(Frobenius Reciprocity) Let H be a subgroup of a group G. Suppose that φ is a character on H and χ is a character on G. Then*

$$[\varphi, \chi_H] = [\varphi^G, \chi].$$

proof. [3, Lemma 5.2].

Lemma 2.6. *(Clifford) Let H be a normal subgroup of a group G and let χ be an irreducible character of G. Let θ be an irreducible constituent of χ_H and suppose that $\theta = \theta_1, \theta_2, \cdots, \theta_t$ are the distinct conjugates of θ in G. Then $\chi_H = e(\theta_1 + \theta_2 + \cdots + \theta_t)$, where $e = [\chi_H, \theta]$.*

proof. [3, Theorem 6.2].

Lemma 2.7. *(Ito) Let G be solvable. Then G has a normal abelian Sylow p-subgroup if and only if every element of c.d.(G) is relative prime to p.*

proof. [3, Theorem 12.33, Corollary 12.34].

Lemma 2.8. *(Schur-Zassenhaus) Let H be a normal S_π-subgroup of G. Then G possesses an $S_{\pi'}$-subgroup which is a complement to H in G.*

proof. [2, Theorem 6.2.1].

3. Main Results

In this section, we prove our main results

Theorem 3.1. *Let B be the intersection of the kernels of all nonlinear irreducible characters of G with π'-degree. Let S be a π-subgroup of G. Then*

$$B \cap G' \cap S \subseteq S'.$$

proof. Suppose that $S_0 = B \cap G' \cap S \not\subseteq S'$. Then there exists a linear character θ of S with $S_0 \not\subseteq \ker\theta$ by Lemma 2.4.

Let χ be an irreducible constituent of the induced character θ^G. Then by Lemma 2.6 (Clifford) we have

$$\chi_S = e(\theta_1 + \theta_2 + \cdots + \theta_t)$$

, where $\theta = \theta_1, \theta_2, \cdots, \theta_t$ are the distinct conjugates of θ in G and $e = [\chi_S, \theta]$. Thus

$$S_0 \not\subseteq \bigcap_{i=1}^{t} \ker\theta_i = \ker\chi_S \subseteq \ker\chi.$$

Note that B is the intersection of the kernels of all nonlinear irreducible characters of G with π'-degree and $S_0 \subseteq B$. Thus it follows that p divides $\chi(1)$ for all nonlinear constituents χ of θ^G for $p \in \pi$.

Since $\theta^G(1) = |G : S| \not\equiv 0 \pmod p$ for all $p \in \pi$, θ^G has a linear constituent μ. In this case, we have $\mu_S = \theta$ by Lemma 2.5 and so

$$S_0 \not\subseteq \ker\theta = S \cap \ker\mu.$$

Thus $S_0 \not\subseteq \ker\mu$. Since $G' \subseteq \ker\mu$, we have

$$B \cap G' \cap S = S_0 \not\subseteq G'$$

, which is a contradiction. Therefore, $B \cap G' \cap S \subseteq S'$.

Theorem 3.2. *Let G be a finite solvable group and let $\pi = \pi$ (c.d.(G)). Then G has a normal π-complement and $G' \cap S = S'$, where S is a S_π-subgroup of G.*

proof. Let $\pi(G) - \pi(\text{c.d.}(G)) = \{p_1, p_2, \cdots, p_r\}$. Then, by **Lemma 2.7 (Ito)** , G has a normal abelian sylow p_i-subgroup S_{p_i} for each $i = 1, 2, \cdots, r$. Thus it follows that the product $S_{p_1} S_{p_2} \cdots S_{p_r}$ is a normal π'-subgroup of G. In this case, Lemma 2.8 yields that G has a S_π-subgroup and so our first assertion holds.

Since $\pi = \pi$ (c.d.(G)), every nonlinear irreducible character has π-degree. Thus the intersection B of all the kernels of all nonlinear irreducible characters of G of π'-degree is G. By Theorem 3.1, we have

$$G' \cap S = B \cap G' \cap S \subseteq S'.$$

Since $S' \subseteq G' \cap S$, we can conclude that $G' \cap S = S'$.

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