韓國數學敎育學會誌<數學敎育>
1992. 12. 제 31권 제 2 호 141-144

Journal of the Korea Society of Mathematical Education
December 1992. Vol. XXXI, No. 2, 141-144

# A PROPERTY OF P-INJECTIVE RING

CHAN YONG HONG

In this paper, some properties of p-injective ring is studied: The Jacobson radical of a p-injective ring which satisfies the ascending chain condition on essential left ideals is nilpotent. Also, the left singular ideal of a ring which satisfies the ascending chain condition on essential left ideals is nilpotent. Finally, we give an example which shows that a semiprime left p-injective ring such that every essential left ideal is two-sided is not necessarily to be strongly regular.

All rings $R$ considered here are associative with identity and all modules are unitary. The Jacobson radical of a ring $R$ will be denoted by $J(R)$, the left singular ideal by $Z(R)$, and the socle of an left $R$-module $M$ by $Soc(M)$. Also, for any subset $X$ of $R$, $l(X)$(resp. $r(X)$) represents the left(resp. right) annihilator of $X$.
A left ideal of $R$ is said to be essential if it has nonzero intersection with each nonzero left ideal of $R$. A nonzero left ideal $U$ of $R$ is said to be uniform if each nonzero left ideal of $R$ in $U$ is essential in $U$. A left $R$-module $M$ has finite Goldie dimension if $M$ does not contain a direct sum of an infinite number of nonzero submodules.

**Lemma 1** [3]. *The following conditions are equivalent:*

(*a*) *A ring $R$ satisfies the ascending chain condition on essential left ideals.*

(*b*) *$R/Soc(R)$ is left Noetherian.*

**Proof.** Suppose that $R$ satisifes the ascending chain condition on essential left ideals. Let $A \subsetneqq B$ be a left ideals of $R$ such that $A$ is essential in $B$. By Zorn's Lemma, there is a left ideal $C$ of $R$ such that $C \cap A = 0$ and $A \oplus C$ is an essential left ideal of $R$. Thus $R/(A \oplus C)$ is left Noetherian. Since $B/A \cong B \oplus C/A \oplus C$, $B/A$ is left Noetherian. Also, every uniform left ideal of $R$ is left Noetherian. Now, let D be a left ideal of $R$ which maximal with respect to the condition $D \cap Soc(R) = 0$. Then $D \oplus Soc(R)$ is essential in $R$ and $R/(D \oplus Soc(R))$ is left Noetherian. Hence, for proving that $R/Soc(R)$ is left Noetherian, it suffices to show that $D$ is left Noetherian. We first show that $D$ has finite Goldie dimension. Assume that $D$ contains an infinite direct

* Associate Professor, Department of Mathematics, Kyung Hee University.

sum $X = X_1 \oplus X_2 \oplus \cdots$ of non-zero left ideals $X_i$. Since $Soc(X_i) = X_i \cap Soc(R) = 0$, each $X_i$ contains a proper essential left ideal $Y_i$ and $Y = Y_1 \oplus Y_2 \cdots$ is essential in $X$. Thus $X/Y$ is left Noetherian. But this is impossible because $X/Y \cong X_1/Y_1 \oplus X_2/Y_2 \oplus \cdots$ with each $X_i/Y_i$ non-zero. This contradiction shows that $D$ has finite Goldie dimension, n say. Then $D$ contains n independent uniform left ideals $U_i$ such that $U = U_1 \oplus U_2 \oplus \cdots$ is essential in $D$. By the above, $U$ and $D/U$ are left Noetherian. The converse is clear.

**Lemma 2 [Levitzki].** *Let $R$ be a left Noetherian ring. Then each nil one sided ideal of $R$ is nilpotent.*

**Proof**. See [ 2 ].

**Theorem 3.** *Let $R$ be a ring which satisfies the ascending chain condition on essential left ideals. Then $Z(R) \subseteqq J(R)$, and $Z(R)$ is nilpotent.*

**Proof**. Let $x \in Z(R)$. Then $l(x)$ is an essential left ideal of R, and so $Soc(R) \subseteqq l(x) \subseteqq l(x^2) \subseteqq \cdots$. Thus there is an integer $k \geqslant 1$ for which $l(x^k) = l(x^{k+j})$ for all integers $j \geqslant 1$. If $ax^k \in Rx^k \cap l(x^k)$ then $ax^{2k} = 0$. Thus $a \in l(x^{2k}) = l(x^k)$, so $ax^k = 0$, i.e., $Rx^k \cap l(x^k) = 0$. Therefore $Rx^k = 0$ since $l(x^k)$ is an essential left ideal of $R$. Thus $x^k = 0$, so $Z(R)$ is a nil ideal of $R$. Hence $Z(R) \subseteqq J(R)$.
Moreover, $(Z(R)+Soc(R))/Soc(R)$ is a nil ideal of $R/Soc(R)$. Since $R\,/\,Soc(R)$ is left Noetherian by Lemma 1, $(Z(R)+Soc(R))/Soc(R)$ is nilpotent by Lemma 2. Thus $(Z(R))^t \subseteqq Soc(R)$ for some integer $t \geqslant 1$. Since $Z(R) \subseteqq J(R)$ and $J(R)$ annihilates all simple left R-modules, $(Z(R))^{t+1} = 0$.

A ring $R$ is called to be left p-injective if for any principal left $P$ of $R$ and left R-module homomorphism $g : P \to R$, there exists $y \in R$ such that $g(b) = by$.

**Lemma 4.** *A ring $R$ is left p-injective if and only if every principal right ideal of $R$ is a right annihilator.*

**Proof.** First, we assume that a ring $R$ is left p-injective. Since a map $f : Ra \to R/l(a)$ defined by $f(xa) = x + l(a)$ is a left $R$-module isomorphism, $Ra \cong R/l(a)$ as left $R$-module. Suppose that $b \in r(l(a))$. Then $l(b) \supseteqq l(r(l(a))) = l(a)$. Thus this induces a left $R$-module epimorphism $R/l(a) \to R/l(b)$ given by $x+l(a) \longmapsto x+l(b)$. Therefore $g : Ra \to Rb$ defined by $g(xa) = xb$ is a left $R$-module epimorphism. Since $R$ is left p-injective, there exists $y \in R$ such that $b = g(a) = ay$. Thus $b \in aR$, so $r(l(a)) \subseteqq aR$. Therefore $aR \subseteqq r(l(aR)) = r(l(a)) \subseteqq aR$, so $r(l(a)) = aR$. Hence $aR$ is a right annihilator.
Conversely, let $h$ be a left $R$-module homomorphism of $Ra$ into $R$. Then $l(a) \subseteqq l(h(a))$ Thus, by hypothesis, $h(a)R = r(l(h(a)R)) \subseteqq r(l(a)) = r(l(aR)) = aR$. Therefore there exists $c \in R$ such that $h(a) = ac$. Hence $R$ is left p-injective.

**Theorem 5.** *Let $R$ be a left p-injective ring which satisfies the ascending chain condition on essential left ideals. Then $J(R)$ is nilpotent.*

**Proof.** Let $x \in J(R)$. Suppose that $t \in R$ with $Rt \cap l(x) = 0$. If $z \in l(tx)$, then $ztx = 0$. Thus $zt \in l(x)$. Since $zt \in Rt \cap l(x) = 0$, $zt = 0$, i.e., $z \in l(t)$. Therefore $l(tx) = l(t)$, i.e., $l(txR) = l(tR)$. By lemma 4, $txR = r(l(txR)) = r(l(tR)) = tR$. Thus $txr = t$ for some $r \in R$, so $t(1 - xr) = 0$. Since $xr \in J(R)$, $1 - xr$ is invertible, and so $t = 0$. Therefore $l(x)$ is an essential left ideal, so $x \in Z(R)$. Thus $J(R) \subseteqq Z(R)$. Since $Z(R)$ is nilpotent by Theorem 3, $J(R)$ is nilpotent.

In fact, $Z(R) = J(R)$ in this case.

**Corollay 6.** *Let $R$ be a left p-injective ring which satisfies the ascending chain condition on essential left ideals. Then every nil left or right ideal is nilpotent.*

Let $Nil^*(R)$ be the upper nil radical of $R$, i.e., $Nil^*R$ is the sum of all nil ideals of $R$. Then $Nil^*R = J(R)$ in this case. Thus if $Nil^*R = 0$, then R has no nonzero nil one-sided ideal, i.e., Kothe conjecture holds for a left p-injective ring which satisfies the ascending chain condition on essential left ideals .

The following proposition is one of other properties of a left p-injective ring.

**Proposition 7.** *Let $R$ be a ring satisfying the condition (*):for left ideals $I$ and $J$ of $R$, $I \cap J = 0$ implies $IJ = 0$. Then the followings are equivalent.*

*a) $R$ is a strongly regular ring.*

*b) $R$ is a semiprime left p-injective ring such that every essential left ideal is two-sided.*

*c) $R$ is a left nonsingular left p-injective ring.*

**Proof.** See [ 4 ].

Related to this Proposition 7, we may ask whether this proposition hold without the condition (*) or not. However, this proposition does not hold without an extra condition. The following example shows that this proposition does not hold without an extra condition such as (*).

**Example 8.** Let $Q$ be a ring of all rational numbers. Let $R = \begin{pmatrix} Q & Q \\ Q & Q \end{pmatrix}$, and $e_{ij}$ $(1 \leqslant i, j \leqslant 2)$ be matrix units. Then $R$ is a semiprime ring. The non-zero left ideals of $R$ are $R$, $Qe_{11} + Qe_{21}$, $Qe_{12} + Qe_{22}$, $I_k = Q(ke_{11} + e_{12}) + Q(ke_{21} + e_{22})$, and $I'_t = Q(e_{11} + te_{12}) + Q(e_{21} + te_{22})$, where $k, t (\neq 0) \in Q$. The essential left ideal is only $R$. Thus every essential left ideal of $R$ is two-sided. The non-zero proper right ideals of $R$ are $Qe_{11} + Qe_{12}$, $Qe_{21} + Qe_{22}$

$J_k = Q(ke_{11} + e_{21}) + Q(ke_{12} + e_{22})$, and $J'_t = Q(e_{11} + te_{21}) + Q(e_{12} + te_{22})$, where $k, t (\neq 0) \in Q$. Since $(Qe_{11} + Qe_{12}) = r(e_{12})$, $(Qe_{21} + Qe_{22}) = r(e_{11})$, $J_k = r(l(ke_{11} + e_{21}))$, and $J'_t = r(l(e_{12} + te_{22}))$, $R$ is a left p-injective. Thus $R$ is a semiprime left p-injective ring such that every essential left ideal is two-sided. But $R$ is not reduced since $e_{12}e_{12} = 0$. Thus $R$ is not strongly regular. $(Qe_{11} + Qe_{12}) \cap (Qe_{21} + Qe_{22}) = 0$, but $(Qe_{11} + Qe_{12})(Qe_{21} + Qe_{22}) \neq 0$. It shows that $R$ does not satisfy the condition (*): for left ideals $I$ and $J$ of $R$, $I \cap J = 0$ implies $IJ = 0$.

## References

[1] E.P. Armendariz, *Rings with dcc on essential left ideals,* J. Comm. Algebra vol 8, 1980, 299–308

[2] A.W. Chatters and C.R. Hajarnavis, *Rings with chain conditions,* Pitman Advanced Publishing Program, Boston, 1980

[3] N.V. Dung, D.V. Huynh and R. Wisbauer, *Quasi-injective modules with acc or dcc on essential submodules,* J. Arch. Math. vol 53, 1989, 252-255

[4] J.Y. Kim and J.K. Park, *On reduced rings and generalized duo ring.*

Department of Mathematics, Kyung Hee University, Seoul, 130-701, Korea