The Journal of the Korean Society of
Coastal and Ocean Engineers
Vol. 4, No. 4, pp. 187~200, December 1992

Lagrangian Motion of Water Particles in Stokes Waves
스토우크스波에서의 水粒子 運動

Tae In Kim* and Im Koo Hwang**
金泰麟* · 黃任九**

Abstract□A general scheme is developed to determine the Langrangian motions of water particles by the Eulerian velocity at their mean positions by using Taylor's theorem. Utilizing the Stokes finite-amplitude wave theory, the orbital motions and the mass transport velocity including the effects of higher-order wave components are determined. The fifth-order approximation of orbital motion gives very good predictions of actual water particle motion in Stokes fifth-order wave theory except near the free-surface. The fifth-order theory predicts the mass transport velocity less than that given by the existing second-order theory over the whole water depth.

要 旨 : Taylor 定理를 이용하여 平均位置에서의 Eulerian 流速으로부터 水粒子의 時間에 따른 Lagrangian 運動軌跡을 決定하는 方法이 提案되었다. 이 方法을 Stokes 有限振幅波理論에 적용하여 高次波成分을 포함하는 水粒子의 運動軌跡과 質量移動速度를 決定하였다. Stokes 5次波理論의 適用 結果, 平均位置에서의 Eulerian 流速으로부터 결정한 水粒子의 運動軌跡은 自由水面 부근을 제외하고는 5次波理論에 의한 瞬間速度에 의해 계산된 값과 매우 좋은 일치를 보여주었다. Stokes 5次波理論에 의한 質量移動速度는 종전의 2次波理論에 의한 質量移動速度보다 全水深에서 작은 값을 보여준다.

1. INTRODUCTION

It is well known that the Lagrangian motion of water particles under wave action results in mass transport in the direction of wave propagation in open channels. In general, the mass transport velocity of a fluid particle is a linear sum of two quantities known as the Stokes drift and the mean Eulerian streaming (Mei, 1983; Craik, 1982). Stokes drift is a general consequence of the irrotational motion of the fluid (Wehausen, 1960) while the mean Eulerian streaming arises due to the viscosity in fluid bounded by a free surface and solid boundaries.

There are two reference coordinate frames that may be used to examine the mass transport due to wave motion: (1) the Eulerian frame which uses a fixed point to observe the mean flux of mass; and (2) the Lagrangian frame which move with the indvidual water particles. The theoretical solutions for the mass transport in progressive waves of permanent-type can be grouped into two main categories. The first one is derived for an ideal fluid by using a finite-amplitude wave theory, while the second results from considering the viscous effects at the bottom and the free-surface boundaries.

In the first category, Stokes (1847) was the first to recognize that in an inviscid, irrotational progressive wave, the fluid particles possess, apart from their orbital motion, a steady drift velocity of $0\{(ak)^2\}$ in the direction of wave progagation ($k=2\pi/L$). The magnitude of this drift is given by

$$u_L = \frac{H^2\,\omega\,k}{8\sinh^2 kh} \cdot \cosh 2k(h+y) \qquad (1)$$

in which u_L is the mass transport velocity in the Lagrangian reference frame; H, ω, and k are the wave height, the wave frequency, and the wave number, respectively, and h is the depth of the water

*陸軍士官學校 土木工學科(Department of Civil Engineering, Korea Military Academy, Seoul 139-799, Korea)
**宇一產業, 江陵市 옥천동 210-090

channel. The vertical distance, y, is measured positive up from the still water level.

The existence of this steady drift under small amplitude progressive waves was proved by Rayleigh (1876) for infinitely deep fluid, and later by Ursell (1953) for finite depth. Skjelbreia (1959) investigated the Lagrangian motion of water particles for Stokes third-order waves. Up to third-order, the Lagrangian mass transport velocity is still given by Eq. (1) with $4a^2$ (a = first-order waves amplitude) instead of H^2. Dalrymple (1976) calculated numerically the mass transport velocity in an Eulerian refernce frame by using the Stream-function wave theory presented by Dean (1974). Wang *et al.* (1982) carried out the same numerical calculation in the Lagrangian reference frame. These inviscid theories for mass transport may be applied with fair accuracy in deep water or during a time not long after the onset of wave motion in which the viscous effects are negligible in core flow of the fluid outside the boundary layer. As time progresses, however, vorticity is generated in the boundary layers at the free surface and at the bottom which then diffuses inward toward the core of the fluid and the mean Eulerian streaming should be added to the Stokes drift.

Several experiments have been conducted in wave flumes in order to confirm the existence of a mean drift velocity profile. The Lagrangian mass transport velocity profiles over the entire water depth were measured by tracing the motion of neutrally buoyant particles. Data have been measured for a flat bottom by Caligny (1878), U.S. Beach Erosion Board (1941), Russell and Osorio (1958), Mei *et al.* (1972), Tsuchiya *et al.* (1980), and for a sloping beach by Bijker *et al.* (1974) and Wang *et al.* (1982).

Measured data show considerable scatter even though the vertical mass transport velocity profiles show some resemblance to Eq. (1) in deep water. The possible reasons for discrepancies between theories and experimental results, and scatter in experiments may be many; such as the existence of higher harmonic waves and beach reflections, the influence of side walls and bottom roughness, and so on.

This research was initiated to extend (1) the Skjelbreia's (1959) 3rd-order mathematical expression of

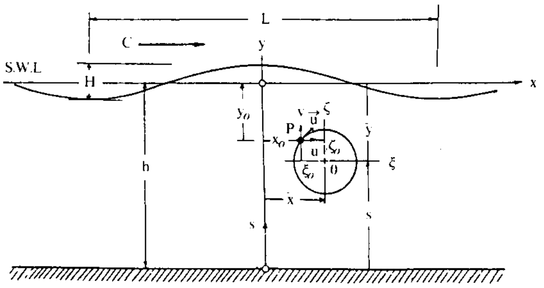


Fig. 1. Definition sketch: Lagrangian motion of water particle under progressive wave of finite amplitude.

the Lagrangin motion of water particles in Stokes wave up to 5th-order, and (2) the existing inviscid mass transport model of Eq. (1) to include the effects of higher-order wave components. Recently, Kim and Hudspeth (1988) presented an extended expression for the mass transport velocity for 5th-order Stokes waves. This paper is a detailed presentation of their extended model, and includes some new findings concerning the behavior of water mass in progressive waves.

2. MATHEMATICAL FORMULATION

Consider a particle initially located at point $P(x_o, y_o)$ at $t=0$ in Fig. 1, and examine the particle over one wave period. It is assumed that the coordinate of the mean particle position, $(\bar{x}, \bar{y})$, is stationary over one wave period. The coordinate (ξ, ζ) represents the particle displacements with respect to the mean particle position. Let $u_L(x_o, y_o, t')$ and $v_L(x_o, y_o, t')$ represent the horizontal and vertical component of the Lagrangian velocity of the particle at time t'.

Let the Lagrangian and Eulerian velocity vectors of the water particles be denoted by $\vec{u}_L$ and $\vec{u}$, respectively, i.e.,

$$\vec{u}_L = u_L \vec{i} + v_L \vec{j};\ \ \vec{u} = u\vec{i} + v\vec{j} \tag{2}$$

The Lagrangian velocity of the particle, $\vec{u}_L$, may be related to the Eulerian velocity, $\vec{u}$, by

$$\vec{u}(x_o, y_o, t) = \vec{u}\left(x_o + \int_0^t u_L(x_o, y_o, t')dt',\ y_o + \int_0^t v_L(x_o, y_o, t')dt', t\right) \tag{3}$$

Using the relationships, $x_o = \bar{x} + \xi_o$, $y_o = \bar{y} + \zeta_o$, and noting that

$$\xi=\xi_o+\int_0^t u_L dt'=\int^t u_L dt',\ \zeta=\zeta_o+\int_0^t v_L dt'=\int^t v_L dt'. \quad (4)$$

the Lagrangian velocity of the water particle is related to the Eulerian velocity at the mean position by

$$\vec{u}(x_o, y_o, t)=\vec{u}(\bar{x}+\int^t u_L(x_o, y_o, t')dt', \bar{y}+\int^t v_L(x_o, y_o, t)dt', t) \quad (5)$$

in which $\int^t u_L(x_o, y_o, t')dt'$ and $\int^t v_L(x_o, y_o, t')dt'$ represent the integrals evaluated at time t, since the values of the integrals evaluated at time 0 are canceled by ξ_o and ζ_o, respectively, in Eq. (4).

By Taylor's theorem, the right hand side of Eq. (5) may be expanded about the mean position $(\bar{x}, \bar{y})$ according to

$$\vec{u}_L(x_o, y_o, t)=\sum_{n=0}^{\infty}\sum_{l=0}^{n}\frac{1}{(n-l)!\,l!}\frac{\partial^n\vec{u}(\bar{x}, \bar{y}, t)}{\partial x^{n-l}\partial y^l}\left[\int^t u_L(x_o, y_o, t')dt'\right]^{n-l}\cdot\left[\int^t v_L(x_o, y_o, t')dt'\right]^l \quad (6)$$

In Eq. (6), the Lagrangian velocity components $u_L(x_o, y_o, t')$ and $v_L(x_o, y_o, t')$ are unknown, and can be related to the Eulerian velocity components at the mean position according to Eq. (6) with t and t' replaced by t' and t'', respectively.

Similarly, the Lagrangian velocity components in the right hand side of the resulting equation may be further related to the Eulerian velocity components at the mean position. By substituting the resulting expressions for the Lagrangian velocity components into Eq. (6) successively, the Lagrangian velocity of the water particle may be related solely to the Eulerian velocity at the mean position to the desired accuracy.

The mass transport velocity may be defined by

$$\bar{u}_L=\frac{1}{T}\int_0^t u_L dt,\ \bar{v}_L=\frac{1}{T}\int_0^t v_L dt \quad (7)$$

in which $T=2\pi/\omega$ being the wave period.

3. ORBITAL MOTION AND MASS TRANSPORT VELOCITY IN STOKES WAVES

3.1 Stokes Wave Theory

For Stokes waves, the Eulerian horizontal and vertical velocity, u and v, are given by

$$u=C\sum_{n=1}^{K} {}_n u; \qquad v=C\sum_{n=1}^{K} {}_n v \quad (8)$$

in which K is the order of Stokes wave theory being considered, and C= the wave celerity of the wave represented by the Kth-order wave theory. The horizontal velocity components take the form

$$\begin{aligned} {}_1u&=F_1 \cosh ks \cos\theta,\ {}_2u=F_2 \cosh 2ks \cos 2\theta \\ {}_3u&=F_{13} \cosh ks \cos\theta+F_3 \cosh 3ks \cos 3\theta \\ {}_4u&=F_{24} \cosh 2ks \cos 2\theta+F_4 \cosh 4ks \cos 4\theta \\ {}_5u&=F_{15} \cosh ks \cos\theta+F_{35} \cosh 3ks \cos 3\theta \\ &\quad +F_5 \cosh 5ks \cos 5\theta \end{aligned} \quad (9)$$

correct to fifth-order. The vertical velocity components take the same form with cosh(·) and cos(·) replaced by sinh(·) and sin(·), respectively. In Eqs. (9), $\theta=kx-\omega t$, $s=y+h$, and k is the wave number.

For the present study, the fifth-order Stokes wave theory presented by Skjelbreia and Hendrickson (1960) is utilized. The coefficients in Eqs. (9) are related with those given by Skjebreia and Herdirickson (1960) as $F_1=\varepsilon A_{11}$, $F_2=2\varepsilon^2 A_{22}$, $F_3=3\varepsilon^3 A_{33}$, $F_{13}=\varepsilon^3 A_{13}$, $F_4=4\varepsilon^4 A_{44}$, $F_{24}=2\varepsilon^4 A_{24}$, $F_5=5\varepsilon^5 A_{55}$, $F_{15}=\varepsilon^5 A_{15}$, $F_{35}=3\varepsilon^5 A_{35}$. Nishmura *et al.* (1977) pointed out a minor mistake in the expression for the fourth-order celerity in the solution given by Skjelbreia and Hendricken (1960) [the sign of $+2592\ C^8$ in the expression of C_2 should be changed into $-2592\ C^8$]. This correction was accounted in calculating the wave number, k, and the perturbation parameter, $\varepsilon(=ka)$, in the present study.

The Lagrangian horizontal and vertical velocity under Kth-order wave, u_L and v_L, are also assumed to be given by

$$u_L=C\sum_{n=1}^{K} {}_n u_L; \qquad v_L=C\sum_{n=1}^{K} {}_n v_L \quad (10)$$

After the Lagrangian velocity components in Eq. (6) are expressed by the Eulerian velocity components by successive expansions, the Eulerian velocity components given by Eq. (8) are substituted into Eq. (6). By retaining the terms up to Kth-order in

Eq. (6), the Lagrangian velocity under a Kth-order Stokes wave may be expressed entirely by the Eulerian quantities. When this procedure is carried out for Stokes wave theory at the different orders, the number of terms in the Eulerian quantity increases rapidly with order. The equations for Lagrangian velocity contain four terms for a Stokes second-order wave, and successively 16, 64, and finally 292 terms for a Stokes fifth-order wave.

3.2 Second-Order Theory

In the Stokes second-order wave theory, the wave height $H=2a$, and the wave celerity is given by $C=C_o \tanh kh$, $C_o=g/\omega$ being the deep water linear wave celerity. The Lagrangian velocity, $\vec{u}_L$, of the water particle is approximated by the Eulerian velocity, $\vec{u}$, according to

$$\vec{u}_L=C({}_1\vec{u}_L+{}_2\vec{u}_L)=C({}_1\vec{u}+{}_1\vec{u}\int^t {}_1u\,dt'+{}_1\vec{u}\int^t {}_1v\,dt') \quad (11)$$

Substitution of Eqs. (9) for ${}_1\vec{u}$ and ${}_2\vec{u}$ into Eq. (11) yields

$$\frac{u_L(x_o,y_o,t)}{C_o}=\frac{Hk}{2\cosh kh}\cosh k\bar{s}\cos\bar{\theta}+\frac{(Hk)^2}{4\sinh 2kh}\left[\frac{3}{2\sinh^2 kh}\cosh 2k\bar{s}-1\right]\cdot\cos 2\bar{\theta}+\frac{(Hk)^2}{4\sinh 2kh}\cosh 2k\bar{s} \quad (12a)$$

and

$$\frac{v_L(x_o,y_o,t)}{C_o}=\frac{Hk}{2\cosh kh}\sinh k\bar{s}\sin\bar{\theta}+\frac{3(Hk)^2\sinh 2k\bar{s}}{16\sinh^3 kh\cdot\cosh kh}\sin 2\bar{\theta} \quad (12b)$$

in which $\bar{s}=\bar{y}+h$, and $\bar{\theta}=k\bar{x}-\omega t$.

Equation (12a) states that the water particles travel, apart from their harmonic velocity components, with a steady-drift velocity of $0\{(Hk)^2\}$ in the direction of wave propagation. Substitution of Eqs. (12) into Eqs. (4) yields

$$\xi=-\frac{H}{2\sinh kh}\cosh k\bar{s}\sin\bar{\theta}-\frac{H^2k}{16\sinh^2 kh}\left[\frac{3}{2\sinh^2 kh}\cosh 2k\bar{s}-1\right]\sin 2\bar{\theta}+(\omega t)\frac{H^2k}{8\sinh^2 kh}\cosh 2k\bar{s} \quad (13a)$$

and

$$\zeta=\frac{H}{2\sinh kh}\sinh k\bar{s}\cos\bar{\theta}+\frac{3H^2k}{32\sinh^4 kh}\sinh 2k\bar{s}\cos 2\bar{\theta} \quad (13b)$$

The components of the mass transport velocity for the Stokes second-order wave given by Eqs. (7) is

$$\frac{\bar{\vec{u}}_L}{C_o}=\frac{(Hk)^2}{8\sinh 2kh}\cosh 2k\bar{s} \quad (14a)$$

and

$$\frac{\bar{v}_L}{C_o}=0 \quad (14b)$$

In deep water, Eq. (14a) reduces to

$$\frac{\bar{\vec{u}}_L}{C_o}=\frac{1}{4}(Hk)^2e^{2k\bar{y}} \quad (15)$$

which was initially reported by Stokes (1847).

3.3 Third-Order Theory

For a Stokes wave at third-order, the first-order wave amplitude, a, and the wave number, k, must be determined for given wave height, H, and angular frequency, ω, from two equations given by

$$\omega^2=gk\tanh kh\,(1+\varepsilon^2 C_1') \quad (16a)$$

$$H=\frac{2}{k}(\varepsilon+\varepsilon^3 B_{33}) \quad (16b)$$

From Eq. (16a), the wave celerity, C, for the third-order is given as $C=C_o \tanh kh(1+\varepsilon^2 C_1')$. The coefficients C_1' and B_{33} given by Skjelbreia and Hendrikson (1960) may be found in Appendix I.

By collecting the terms up to third-order in Eq. (6), the Lagrangian velocity components are given by

$$\frac{u_L(x_o,y_o,t)}{C_o}=[1+\varepsilon^2 C_1']\tanh kh\left\{\left[\left(F_1+F_{13}-\frac{3}{8}F_1^3\right)\cosh k\bar{s}+\frac{F_1}{8}(F_1^2+10F_2)\cosh 3k\bar{s}\right]\cos\bar{\theta}\right.$$

$$+\left(F_2\cosh 2k\bar{s}-\frac{F_1^2}{2}\right)\cos 2\bar{\theta}$$

$$+\left[\frac{F_1}{4}(F_1^2-5F_2)\cosh k\bar{s}+F_3\cosh 3k\bar{s}\right]\cos 3\bar{\theta}$$

$$+\frac{F_1^2}{2}\left[\cosh 2k\bar{s}-(\omega t)F_1\cosh k\bar{s}\cosh 2k\bar{s}\sin\theta\right]\Big\} \tag{17a}$$

$$\frac{v_L(x_o,y_o,t)}{C_o}=[1+\varepsilon^2 C_1']\tanh kh$$

$$\Big\{\left[\left(F_1+F_{13}-\frac{F_1^3}{8}\right)\sinh k\bar{s}\right.$$

$$\left.+\frac{F_1}{8}(6F_2-F_1^2)\sinh 3k\bar{s}\right]\sin\bar{\theta}+F_2\sinh 2k\bar{s}\cdot\sin 2\bar{\theta}$$

$$+(F_3\sinh 3k\bar{s}-\frac{3}{4}F_1F_2\sinh k\bar{s})\sin 3\bar{\theta}$$

$$+(\omega t)\frac{F_1^3}{2}\sinh k\bar{s}\cdot\cosh 2k\bar{s}\cos\bar{\theta}\Big\} \tag{17b}$$

The horizontal and vertical displacements of the water particles from their mean positions, ξ and ζ, determined by Eqs. (4), are given by

$$\xi=-\frac{1}{k}\left[\left(F_1+F_{13}-\frac{F_1^3}{8}\right)\cosh k\bar{s}\right.$$

$$\left.+\frac{F_1}{8}(3F_1^2+10F_2)\cosh 3k\bar{s}\right]\sin\bar{\theta}$$

$$-\frac{1}{2k}\left[F_2\cosh 2k\bar{s}-\frac{F_1^2}{2}\right]\sin 2\bar{\theta}$$

$$-\frac{1}{3k}\left[\frac{F_1}{4}(F_1^2-5F_2)\cosh k\bar{s}+F_3\cosh 3k\bar{s}\right]\sin 3\bar{\theta}$$

$$+\frac{Ct}{2}F_1^2(\cosh 2k\bar{s}-F_1\cosh k\bar{s}\cdot\cosh 2k\bar{s}\cdot\cos\theta) \tag{18a}$$

and

$$\zeta=\frac{1}{k}\left[\left(F_1+F_{13}-\frac{3F_1^3}{8}\right)\sinh k\bar{s}\right.$$

$$\left.+\frac{F_1}{8}(F_1^2+6F_2)\sinh 3k\bar{s}\right]\cos\bar{\theta}$$

$$+\frac{1}{2k}F_2\sinh 2k\bar{s}\cdot\cos 2\bar{\theta}$$

$$+\frac{1}{3k}\left[F_3\sinh 3k\bar{s}-\frac{3}{4}F_1F_2\sinh k\bar{s}\right]\cos 3\bar{\theta}$$

$$-\frac{Ct}{2}F_1^3\sinh k\bar{s}\cdot\cosh 2k\bar{s}\cdot\sin\bar{\theta} \tag{18b}$$

The third-order Lagrangian velocity has been reported by Skjelbreia (1959) by use of the Taylor series expansion scheme in a different manner. He substituted $\bar{x}+\xi$ for x and $\bar{s}+\xi$ for s in Eqs. (9) for the horizontal and vertical components of the orbital velocities and expanded. The expressions for horizontal and vertical particle positions, ξ and ζ, are found by successive approximations. Equations (17) and (18) are identical to the expressions reported by Skjelbreia (1959) [also quoted by Wiegel (1964)] provided that his expression $3F_1^3$ in $\sin\bar{\theta}$ term in Eq. (18a) is changed into $3F_1^2$. This seems to be a typogrpahical error, for the term in question should be a quantity of $O(\varepsilon^2)$. In addition, Skjelbreia (1959) missed F_{13} into his expression for both $\sin\bar{\theta}$ term in Eq. (18a) and $\cos\bar{\theta}$ term in Eq. (18b), which should be included to be mathematically consistent.

The components of the mass transport velocity for a Stokes wave at third-order according to the definition given by Eqs. (7) is determined to be

$$\frac{u_L}{C_o}=(1+\varepsilon^2 C_1')\tanh kh\left[\frac{F_1^2}{2}\cosh 2ks\right.$$

$$\left.-\frac{F_1^3}{4}(\cosh ks+\cosh 3ks)\cos k\bar{x}\right] \tag{19a}$$

and

$$\frac{v_L}{C_o}=\{1+\varepsilon^2 C_1'\}\tanh kh$$

$$\left[\frac{F_1^3}{4}(\sinh ks-\sinh 3ks)\sin k\bar{x}\right] \tag{19b}$$

Equations (19) indicate that mass transport velocity has a component of $O(\varepsilon^3)$ which is dependent on x, and that the vertical mass transport velocity is nonzero at third or higher-orders. However, the second term in Eq. (19a) as well as the vertical mass transport velocity tends to average to zero as a water particle travels over a distance equal to one wavelength, so that the long-term mass transport velocity reduces to

$$\frac{u_L}{C_o}=(1+\varepsilon^2 C_1')\tanh kh\frac{F_1^2}{2}\cosh 2ks \tag{20}$$

In deep water, Eq. (20) reduces to

$$\frac{u_L}{C_o}=[1+\frac{1}{2}(ak)^2](ak)^2e^{2ky} \tag{21}$$

3.4 Fourth-Order Theory

For the Stokes fourth-order wave theory, the first-order wave amplitude, a, the wave number, k, and wave celerity, C, are identical as determined for the Stokes *third-order wave. By collecting terms up to* fourth-order in Eq. (6), the Lagrangian velocity is given by

$$\frac{u_L(x_o, y_o, t)}{C_o} = [1+\varepsilon^2 C_1'] \tanh kh \Big\{ \Big[\Big(F_1 + F_{13} - \frac{3}{8} F_1^3 \Big) \cosh k\bar{s} + \frac{F_1}{8}(F_1^2 + 10F_2)\cosh 3k\bar{s} \Big] \cos\bar{\theta}$$
$$+ \Big[\frac{F_1}{48}(10F_1^3 - 5F_1F_2 - 48F_{13} - 24F_1) - \frac{1}{24}(4F_1^4 + 37F_1^2F_2 - 24F_{24} - 24F_2)\cosh 2k\bar{s} - \frac{F_1}{48}(2F_1^3 - 3F_1F_2 - 80F_3)\cosh 4k\bar{s} \Big] \cos 2\bar{\theta}$$
$$+ \Big[\frac{F_1}{4}(F_1^2 - 5F_2)\cosh k\bar{s} + F_3 \cosh 3k\bar{s} \Big] \cos 3\bar{\theta}$$
$$- \Big[\frac{1}{96}(5F_1^4 - 61F_1^2F_2 + 48F_2^2) + \frac{F_1}{96}(F_1^3 - 34F_1F_2 + 160F_3)\cosh 2k\bar{s} + \frac{1}{32}(F_1^2F_2 - 32F_4)\cosh 4k\bar{s} \Big] \cos 4\bar{\theta}$$
$$- \frac{F_1^3}{4}(\cosh k\bar{s} + \cosh 3k\bar{s})\ (\omega t)\ \sin\theta$$
$$- \Big[\frac{F_1^2}{16}(F_1^2 + 8F_2) - \frac{3}{8}F_1^4 \cosh 2k\bar{s} + \frac{F_1^2}{16}(F_1^2 + 8F_2)\cosh 4k\bar{s} \Big] (\omega t)\ \sin 2\theta$$
$$+ \frac{1}{32} \Big[F_1^2(3F_1^2 - F_2) - F_1(5F_1^3 + 10F_1F_2 - 32F_{13} - 16F_1)\ \cosh 2k\bar{s} + (4F_1^4 + 31F_1^2F_2 + 16F_2^2)\cosh 4k\bar{s} \Big] \Big\} \qquad (22a)$$

and

$$\frac{v_L(x_o, y_o, t)}{C_o} = [1+\varepsilon^2 C_1'] \tanh kh \Big\{ \Big[(F_1 + F_{13} - F_1^3) \sinh k\bar{s} - \frac{F_1}{8}(F_1^2 - 6F_2)\sinh 3k\bar{s} \Big] \sin\theta$$
$$+ \Big[\frac{1}{48}(5F_1^4 - 16F_1^2F_2 + 48F_2 + 48F_{24})\sinh 2k\bar{s} - \frac{F_1}{12}(F_1^3 + 3F_1F_2 - 16F_3)\sinh 4k\bar{s} \Big] \sin 2\bar{\theta}$$
$$+ \Big[F_3 \sinh 3k\bar{s} - \frac{3}{4}F_1F_2 \sinh k\bar{s} \Big] \sin 3\bar{\theta}$$
$$+ \Big[\frac{F_1}{96}(3F_1^3 + 16F_1F_2 - 128F_3)\sinh 2k\bar{s} + F_4 \sinh 4k\bar{s} \Big] \sin 4\bar{\theta}$$
$$+ (\omega t)\frac{F_1^3}{2}\sinh k\bar{s}\cdot\cosh 2k\bar{s}\cdot\cos\bar{\theta}$$
$$+ (\omega t)\frac{F_1^2}{16}(F_1^2 + 8F_2)\sinh 4k\bar{s}\cdot\cos 2\bar{\theta} \Big\} \qquad (22b)$$

The horizontal and vertical displacement of particles are now given by

$$\xi = -\frac{1}{k}\Big[\Big(F_1 + F_{13} - \frac{F_1^3}{8} \Big) \cosh k\bar{s} + \frac{F_1}{8}(3F_1^2 + 10F_2)\cosh 3k\bar{s} \Big] \sin\bar{\theta}$$
$$- \frac{1}{2k}\Big[\frac{F_1}{96}(23F_1^3 + 14F_1F_2 - 96F_{13} - 48F_1) - \frac{1}{48}(17F_1^4 + 74F_1^2F_2 - 48F_{24} - 48F_2)\cosh 2k\bar{s} - \frac{F_1}{96}(F_1^3 - 30F_1F_2 - 160F_3)\cosh 4k\bar{s} \Big] \sin 2\bar{\theta}$$
$$- \frac{1}{3k}\Big[\frac{F_1}{4}(F_1^2 - 5F_2)\cosh k\bar{s} + F_3 \cosh 3k\bar{s} \Big] \sin 3\bar{\theta}$$
$$+ \frac{1}{4k}\Big[\frac{1}{96}(5F_1^4 - 61F_1^2F_2 + 48F_2^2) + \frac{F_1}{96}(F_1^3 - 34F_1F_2 + 160F_3)\cosh 2k\bar{s} - \frac{1}{32}(F_1^2F_2 - 32F_4)\cosh 4k\bar{s} \Big] \sin 4\bar{\theta}$$
$$- \frac{Ct}{32}\Big\{ 8F_1^3(\cosh k\bar{s} + \cosh 3k\bar{s})\cos\bar{\theta} + [F_1^2(F_1^2 + 8F_2) - 6F_1^4 \cos 2k\bar{s} + F_1^2(F_1^2 + 8F_2)\cosh 4k\bar{s}]\cos 2\bar{\theta} \Big\}$$
$$+ \frac{Ct}{32}[F_1^2(3F_1^2 - F_2) - F_1(5F_1^3 + 10F_1F_2 - 32F_{13} - 16F_1)\cosh 2k\bar{s} + (4F_1^4 + 31F_1^2F_2 + 16F_2^2)\cosh 4k\bar{s}] \qquad (23a)$$

and

$$\zeta = \frac{1}{k}\Big[\Big(F_1 + F_{13} - \frac{3}{8}F_1^3 \Big) \sinh k\bar{s} + \frac{F_1}{8}(F_1^2 + 6F_2)\sinh 3k\bar{s} \Big] \cos\bar{\theta}$$

$$+\frac{1}{2k}\left[\frac{1}{48}(5F_1^4-16F_1^2F_2+48F_2+48F_{24})\sinh 2k\bar{s} - \frac{F_1}{96}(5F_1^3-128F_3)\sinh 4k\bar{s}\right]\cos 2\bar{\theta}$$
$$+\frac{1}{3k}\left[-\frac{3}{4}F_1F_2\sinh k\bar{s}+F_3\sinh 3k\bar{s}\right]\cos 3\bar{\theta}$$
$$+\frac{1}{4k}\left[\frac{F_1}{96}(3F_1^3+16F_1F_2-128F_3)\sinh 2k\bar{s} + F_4\sinh 4k\bar{s}\right]\cos 4\bar{\theta}$$
$$+\frac{Ct}{32}\left[8F_1^3(\sinh k\bar{s}-\sinh 3k\bar{s})\sin\bar{\theta} - F_1^2(F_1^2+8F_2)\sinh 4k\bar{s}\cdot\sin 2\bar{\theta}\right] \quad (23b)$$

Accordinng to Eqs. (7), the components of the mass transport velocity for the fourth-order Stokes wave are determined as follows:

$$\frac{\bar{u}_L}{C_o}=(1+\varepsilon^2C_1')\tanh kh\left\{\frac{1}{32}[F_1^2(3F_1^2-F_2) - F_1(5F_1^3+10F_1F_2-32F_{13}-16F_1)\cosh 2k\bar{s} + (4F_1^4+31F_1^2F_2+16F_2^2)\cosh 4k\bar{s}] - \frac{F_1^3}{4}(\cosh k\bar{s}+\cosh 3k\bar{s})\cos k\bar{x} - \frac{1}{32}[F_1^2(F_1^2+8F_2)-6F_1^4\cosh 2k\bar{s} + F_1^2(F_1^2+8F_2)\cosh 4k\bar{s}]\cos 2k\bar{x}\right\} \quad (24a)$$

and

$$\frac{\bar{v}_L}{C_o}=(1+\varepsilon^2C_1')\tanh kh\left[\frac{F_1^3}{4}(\sinh k\bar{s}-\sinh 3k\bar{s})\cdot\sin k\bar{x} - \frac{F_1^2}{32}(F_1^2+8F_2)\sinh 4k\bar{s}\cdot\sin 2k\bar{x}\right] \quad (24b)$$

The terms dependent on $\bar{x}$ in Eqs. (24) tend to average to zero as a water particle travels over a distance equal to one wavelength, so that the long-term mass transport velocity may be given by

$$\frac{\bar{u}_L}{C_o}=(1+\varepsilon^2C_1')\tanh kh\left\{\frac{1}{32}[F_1^2(3F_1^2-F_2) - F_1(5F_1^3+10F_1F_2-32F_{13}-16F_1)\cosh 2k\bar{s} + (4F_1^4+31F_1^2F_2+16F_2^2)\cosh 4k\bar{s}]\right\} \quad (25)$$

In deep water, Eq. (25) reduces to

$$\frac{\bar{u}_L}{C_o}=\left[1+\frac{1}{2}(ak)^2\right]\cdot\left[(ak)^2e^{2k\bar{y}}-\frac{5}{4}(ak)^4e^{2k\bar{y}}+(ak)^4e^{4k\bar{y}}\right] \quad (26)$$

3.5 Fifth-Order Theory

In the Stokes fifth-order wave theory, the first-order wave amplitude, a, and the wave number, k, must be determined from two equations given by

$$\omega^2=gk\tanh kh\,(1+\varepsilon^2C_1'+\varepsilon^4C_2') \quad (27a)$$

and

$$H=\frac{2}{k}(\varepsilon+\varepsilon^3B_{33}+\varepsilon^5B_{55}) \quad (27b)$$

From Eq. (27a), the wave celerity, C, for the fifth-order wave is given as $C=C_o\tanh kh(1+\varepsilon^2C_1'+\varepsilon^4C_2')$. The coefficients B_{35} and B_{55} given by Skjelbreia and Hendrickson (1960) may be found in Appendix I.

By collecting the terms up to fifth-order in Eq. (6), the Lagrangian velocity is given by

$$\frac{u_L(x_o,y_o,t)}{C_o}=(1+\varepsilon^2C_1'+\varepsilon^4C_2')\tanh kh$$
$$\{(U_{11}\cosh k\bar{s}+U_{13}\cosh 3k\bar{s}+U_{15}\cosh 5k\bar{s})\cos\bar{\theta}$$
$$+(U_{20}+U_{22}\cosh 2k\bar{s}+U_{24}\cosh 4k\bar{s})\cdot\cos 2\bar{\theta}$$
$$+(U_{31}\cosh k\bar{s}+U_{33}\cosh 3k\bar{s}+U_{35}\cosh 5k\bar{s})\cdot\cos 3\bar{\theta}$$
$$+(U_{40}+U_{42}\cosh 2k\bar{s}+U_{44}\cosh 4k\bar{s})\cos 4\bar{\theta}$$
$$+(U_{51}\cosh k\bar{s}+U_{53}\cosh 3k\bar{s}+U_{55}\cosh 5k\bar{s})\cosh 5\bar{\theta}$$
$$+(S_{11}\cosh k\bar{s}+S_{13}\cosh 3k\bar{s}+S_{15}\cosh 5k\bar{s})(\omega t)\sin\bar{\theta}$$
$$+(S_{20}+S_{22}\cosh 2k\bar{s}+S_{24}\cosh 4k\bar{s})(\omega t)\sin 2\bar{\theta}$$
$$+(S_{31}\cosh k\bar{s}+S_{33}\cosh 3k\bar{s}+S_{35}\cosh 5k\bar{s})(\omega t)\sin 3\bar{\theta}$$
$$-\frac{F_1^5}{64}(2\cosh k\bar{s}+5\cosh 3k\bar{s}+\cosh 5k\bar{s})(\omega t)^2\cos\bar{\theta}$$
$$+\frac{1}{32}[F_1^2(3F_1^2-F_2) - F_1(5F_1^3+10F_1F_2-32F_{13}-16F_1)\cos 2k\bar{s} + (4F_1^4+31F_1^2F_2+16F_2^2)\cdot\cosh 4k\bar{s}]\} \quad (28a)$$

in which

$U_{11}=[1536(F_1+F_{13}+F_{15})-576F_1^3-1872F_1^2F_{13}+264F_1^2F_3+1696F_1^3F_2+160F_1F_2^2+65F_1^5]/1536,$

$U_{13}=(384F_1^3+3840F_1F_2+1440F_1^2F_{13}+3840F_1F_{24}+3840F_2F_{13}-1552F_1^2F_3-5276F_1^3F_2-2608F_1F_2^2-383F_1^5)/3072,$

$U_{15}=(3328F_2F_3+4864F_1^2F_3+1756F_1^3F_2+4656F_1F_2^2$

$$+83F_1^5)/3072,$$
$$U_{20}=F_1(10F_1^3-5F_1F_2-48F_{13}-24F_1)/48,$$
$$U_{22}=-(4F_1^4+37F_1^2F_2-24F_2-24F_{24})/24,$$
$$U_{24}=-F_1(2F_1^3-3F_1F_2-80F_3)/48,$$
$$U_{31}=(768F_1^3-3840F_1F_2+2304F_1^2F_{13}-3840F_1F_{24}$$
$$-3840F_2F_{13}-192F_1^2F_3+3012F_1^2F_2-4896F_1F_2^2$$
$$-131F_1^5)/3072,$$
$$U_{33}=[6144(F_3+F_{35})-15040F_1^2F_3+2124F_1^3F_2$$
$$+96F_1F_2^2+531F_1^5]/6144,$$
$$U_{35}=F_1(13056F_4-512F_1F_3-476F_1^2F_2-1632F_2^2$$
$$+179F_1^4)/6144,$$
$$U_{40}=-(5F_1^4-61F_1^2F_2+48F_2^2)/96,$$
$$U_{42}=-F_1(F_1^3-34F_1F_2+160F_3)/96,$$
$$U_{44}=-(F_1^2F_2-32F_4)/32,$$
$$U_{51}=(4064F_1^2F_3-3328F_2F_3-2156F_1^3F_2+2656F_1F_2^2$$
$$+175F_1^5)/3072,$$
$$U_{53}=F_1(2624F_1F_3-13056F_4+340F_1^2F_2+512F_2^2$$
$$+117F_1^4)/6144,$$
$$U_{55}=(6144F_5+256F_1^2F_3+612F_1^3F_3+21F_1^5)/6144,$$
$$S_{11}=-F_1^2(64F_1+80F_1F_2+192F_{13}+21F_1^3)/256,$$
$$S_{13}=-F_1(128F_1^2+136F_1^2F_2+128F_2^2+384F_1F_{13}$$
$$-47F_1^4)/512,$$
$$S_{15}=-F_1(200F_1^2F_2+128F_2^2+53F_1^4)/512,$$
$$S_{20}=-F_1^2(F_1^2+8F_2)/16,$$
$$S_{22}=3F_1^4/8,\quad S_{24}=-F_1^2(F_1^2+8F_2)/16,$$
$$S_{31}=-F_1^2(48F_3-56F_1F_2+13F_1^3)/64,$$
$$S_{33}=-F_1^3(21F_1^2-124F_2)/128,$$
$$S_{35}=-F_1^2(96F_3-4F_1F_2+F_1^3)/128,$$

and

$$\frac{v_L(x_o,y_o,t)}{C_o}=(1+\varepsilon^2C_1'+\varepsilon^4C_2')\tanh kh\{(V_{11}\sinh k\bar{s}$$
$$+V_{13}\sinh 3k\bar{s}+V_{15}\sinh 5k\bar{s})\sin\bar{\theta}$$
$$+(V_{22}\sinh 2k\bar{s}+V_{24}\sinh 4k\bar{s})\sin 2\bar{\theta}$$
$$+(V_{31}\sinh k\bar{s}+V_{33}\sinh 3k\bar{s}+V_{35}\sinh 5k\bar{s})\sin 3\bar{\theta}$$
$$+(V_{42}\sinh 2k\bar{s}+V_{44}\sinh 4k\bar{s})\sin 4\bar{\theta}$$
$$+(V_{51}\sinh k\bar{s}+V_{53}\sinh 3k\bar{s}+V_{55}\sinh 5k\bar{s})\sin 5\bar{\theta}$$
$$+(T_{11}\sinh k\bar{s}+T_{13}\sinh 3k\bar{s}+T_{15}\sinh 5k\bar{s})(\omega t)\cos\bar{\theta}$$
$$+T_{24}(\omega t)\sinh 4k\bar{s}\cdot\cos 2\bar{\theta}$$
$$+(T_{31}\sinh k\bar{s}+T_{33}\sinh 3k\bar{s}+S_{35}\sinh 5k\bar{s})$$
$$(\omega t)\cos 3\bar{\theta}+\frac{F_1^5}{32}(3\sinh k\bar{s}-\sinh 3k\bar{s})(\omega t)^2\sin\bar{\theta}\} \quad (28b)$$

in which

$$V_{11}=[1536(F_1+F_{13}+F_{15})-192F_1^3-64F_1^2F_3+64F_1^3F_2$$
$$-352F_1F_2^2+257F_1^5-288F_1^2F_{13}]1536,$$
$$V_{13}=(-384F_1^3+2304F_1F_2+1536F_1^2F_3-1736F_1^3F_2$$
$$-512F_1F_2^2+165F_1^5+2304F_2F_{13}+2304F_1F_{24}$$
$$-576F_1^2F_{13})/3072,$$
$$V_{15}=(1280F_2F_3+2432F_1^2F_3-1096F_1^3F_2+1536F_1F_2^2$$
$$-331F_1^5)/3072,$$
$$V_{22}=(5F_1^4-16F_1^2F_2+48F_2+48F_{24})/48,$$
$$V_{24}=-F_1(F_1^3+3F_1F_2-16F_3)/12,$$
$$V_{31}=(3072F_{15}-2304F_1F_2+816F_1^2F_3+884F_1^3F_2$$
$$+1728F_1F_2^2-2304F_1F_{24}-2304F_2F_{13}-147F_1^5)/3072,$$
$$V_{33}=(6144F_3+6144F_{35}-12192F_1^2F_3-260F_1^3F_2$$
$$+96F_1F_2^2+63F_1^5)/6144,$$
$$V_{35}=F_1(11520F_4-256F_1F_3-1900F_1^2F_2-1632F_2^2$$
$$+85F_1^4)/6144,$$
$$V_{42}=F_1(3F_1^3+16F_1F_2-128F_3)/96,\quad V_{44}=F_4,$$
$$V_{51}=(2816F_1^2F_3-1280F_2F_3-366F_1^3F_2-320F_1F_2^2$$
$$-17F_1^5)/3072,$$
$$V_{53}=F_1(1568F_1^2F_3-11520F_4+292F_1^2F_2+416F_2^2$$
$$+F_1^4)/6144,$$
$$V_{55}=(6144F_5+288F_1^2F_3+320F_1^3F_2+96F_1F_2^2$$
$$+35F_1^5)/6144,$$
$$T_{11}=F_1^2(36F_1F_2+3F_1^3-96F_{13}-32F_1)/128,$$
$$T_{13}=F_1(192F_1F_{13}-224F_1^2F_2-64F_2^2-21F_1^4$$
$$+64F_1^2)/256,$$
$$T_{15}=F_1(184F_1^2F_2+64F_2^2+5F_1^4)/256,$$
$$T_{24}=F_1^2(F_1^2+8F_2)/16,$$
$$T_{31}=F_1^2(48F_3+34F_1F_2+5F_1^3)/64,$$
$$T_{33}=-F_1^3(39F_2+2F_1^2)/64,$$
$$T_{35}=F_1^2(48F_3-F_1F_2+F_1^3)/64.$$

The horizontal and vertical displacement of water particles from their mean positions, ξ and ζ, determined by Eqs. (4) are given by

$$\xi=-\frac{1}{k}(U_{11}'\cosh k\bar{s}+U_{13}'\cosh 3k\bar{s}+U_{15}'\cosh 5k\bar{s})\sin\bar{\theta}$$
$$-\frac{1}{2k}(U_{20}'+U_{22}'\cosh 2k\bar{s}+U_{24}'\cosh 4k\bar{s})\sin 2\bar{\theta}$$
$$-\frac{1}{3k}(U_{31}'\cosh k\bar{s}+U_{33}'\cosh 3k\bar{s}+U_{35}'\cosh 5k\bar{s})\sin 3\bar{\theta}$$
$$-\frac{1}{4k}(U_{40}'+U_{42}'\cosh 2k\bar{s}+U_{44}'\cosh 4k\bar{s})\sin 4\bar{\theta}$$
$$-\frac{1}{5k}(U_{51}'\cosh k\bar{s}+U_{53}'\cosh 3k\bar{s}+U_{55}'\cosh 5k\bar{s})\sin 5\bar{\theta}$$
$$-(Ct)(S_{11}'\cosh k\bar{s}+S_{13}'\cosh 3k\bar{s}+S_{15}'\cosh 5k\bar{s})\cos\bar{\theta}$$
$$-\frac{(Ct)}{2}(S_{20}'+S_{22}'\cosh 2k\bar{s}+S_{24}'\cosh 4k\bar{s})\cos 2\bar{\theta}$$
$$-\frac{(Ct)}{3}(S_{31}'\cosh k\bar{s}+S_{33}'\cosh 3k\bar{s}+S_{35}'\cosh 5k\bar{s})\cos 3\bar{\theta}$$

$$+k(Ct)^2\frac{F_1^5}{64}(2\cosh k\bar{s}+5\cosh 3k\bar{s}+\cos 5k\bar{s})\sin\theta$$
$$+\frac{(Ct)}{32}[F_1^2(3F_1^2-F_2)-F_1(5F_1^3+10F_1F_2-32F_{13}$$
$$-16F_1)\cosh 2k\bar{s}+(4F_1^4+31F_1^2F_2+16F_2^2)\cosh 4k\bar{s} \quad (29a)$$

in which

$U_{11}'=[1536(F_1+F_{13}+F_{15})-192F_1^3-720F_1^2F_{13}+264F_1^2F_3+2176F_1^3F_2+160F_1F_2^2+287F_1^5]/1536,$
$U_{13}'=(1152F_1^3+3840F_1F_2+3744F_1^2F_{13}+3840F_1F_{24}+3840F_2F_{13}-1552F_1^2F_3-4460F_1^3F_2-1840F_1F_2^2-185F_1^5)/3072,$
$U_{15}'=(3328F_2F_3+4864F_1^2F_3+2956F_1^3F_2+5424F_1F_2^2+331F_1^5)/3072,$
$U_{20}'=F_1(23F_1^3+14F_1F_2-96F_{13}-48F_1)/96,$
$U_{22}'=-(17F_1^4+74F_1^2F_2-48F_2-48F_{24})/48,$
$U_{24}'=-F_1(F_1^3-30F_1F_2-160F_3)/96,$
$U_{31}'=(768F_1^3-3840F_1F_2+2304F_1^2F_{13}-3840F_1F_{24}-3840F_2F_{13}+576F_1^2F_3+2116F_1^3F_2-4896F_1F_2^2+77F_1^5)/3072,$
$U_{33}'=(6144F_3+6144F_{35}-15040F_1^2F_3+140F_1^3F_2+96F_1F_2^2+867F_1^5)/6144,$
$U_{35}'=F_1(13056F_4+1024F_1F_3-540F_1^2F_2-1632F_2^2+195F_1^4)/6144,$
$U_{40}'=U_{40},\ U_{42}'=U_{42},\ U_{44}'=U_{44},\ U_{51}'=U_{51},\ U_{53}'=U_{53},\ U_{55}'=U_{55},$
$S_{11}'=F_1^2(64F_1+80F_1F_2+192F_{13}+37F_1^3)/256,$
$S_{13}'=F_1(128F_1^2+136F_1^2F_2+128F_2^2+384F_1F_{13}+33F_1^4)/512,$
$S_{15}'=F_1(200F_1^2F_2+128F_2^2+69F_1^4)/512,\ S_{20}'=-S_{20},$
$S_{22}'=-S_{22},\ S_{24}'=-S_{24},\ S_{31}'=-S_{31},\ S_{33}'=-S_{33},\ S_{35}'=-S_{35},$

and

$$\zeta=\frac{1}{k}(V_{11}'\sinh k\bar{s}+V_{13}'\sinh 3k\bar{s}+V_{15}'\sinh 5k\bar{s})\cos\bar{\theta}$$
$$+\frac{1}{2k}(V_{22}'\sinh 2k\bar{s}+V_{24}'\sinh 4k\bar{s})\cos 2\bar{\theta}$$
$$+\frac{1}{k}(V_{31}'\sinh k\bar{s}+V_{33}'\sinh 3k\bar{s}+V_{35}'\sinh 5k\bar{s})\cos 3\bar{\theta}$$
$$+\frac{1}{4k}(V_{42}'\sinh 2k\bar{s}+V_{44}'\sinh 4k\bar{s})\cos 4\bar{\theta}$$
$$+\frac{1}{5k}(V_{51}'\sinh k\bar{s}+V_{53}'\sinh 3k\bar{s}+V_{55}'\sinh 5k\bar{s})\cos 5\bar{\theta}$$
$$-(Ct)(T_{11}'\sinh k\bar{s}+T_{13}'\sin 3k\bar{s}+T_{15}'\sin 5k\bar{s})\sin\bar{\theta}$$
$$-\frac{(Ct)}{2}T_{24}'\sinh 4k\bar{s}\cdot\sin 2\bar{\theta}$$
$$-\frac{(Ct)}{3}(T_{31}'\sinh k\bar{s}+T_{33}'\sinh 3k\bar{s}+T_{35}'\sinh 5k\bar{s})\sin 3\bar{\theta}$$
$$+k(Ct)^2\frac{F_1^5}{32}(3\sinh k\bar{s}-\sinh 3k\bar{s})\cos\bar{\theta} \quad (29b)$$

in which

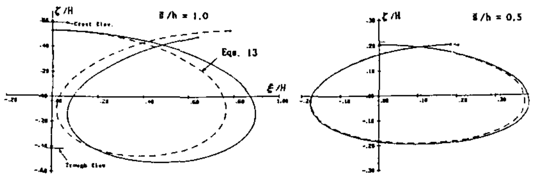
$V_{11}'=[1536(F_1+F_{13}+F_{15})-576F_1^3-64F_1^2F_3+496F_1^3F_2-352F_1F_2^2+5F_1^5-1440F_1^2F_{13})]/1536,$
$V_{13}'=(384F_1^3+2304F_1F_2+1536F_1^2F_3-4424F_1^3F_2-1280F_1F_2^2+105F_1^5+2304F_2F_{13}+2304F_1F_{24}+1728F_1^2F_{13})/3072,$
$V_{15}'=(1280F_2F_3+2432F_1^2F_3+1112F_1^3F_2+2304F_1F_2^2-271F_1^5)/3072,$
$V_{22}'=V_{22},\ V_{24}'=-F_1(5F_1^3-128F_3)/96,$
$V_{31}'=(3072F_{15}-2304F_1F_2+1584F_1^2F_3+1428F_1^3F_2+1728F_1F_2^2-2304F_1F_{24}-2304F_2F_{13}-67F_1^5)/3072,$
$V_{33}'=(6144F_3+6144F_{35}-12192F_1^2F_3-1508F_1^3F_2+96F_1F_2^2-F_1^5)/6144,$
$V_{35}'=F_1(11520F_4+1280F_1F_3-1932F_1^2F_2-1632F_2^2+117F_1^4)/6144,$
$V_{42}'=V_{42},\ V_{44}'=V_{44},\ V_{51}'=V_{51},\ V_{53}'=V_{53},\ V_{55}'=V_{55},$
$T_{11}'=F_1^2(36F_1F_2-21F_1^3-96F_{13}-32F_1)/128,$
$T_{13}'=F_1(192F_1F_{13}-224F_1^2F_2-64F_2^2-5F_1^4+64F_1^2)/256,$
$T_{15}'=T_{15},\ T_{24}'=T_{24},\ T_{31}'=T_{31},\ T_{33}'=T_{33},\ T_{35}'=T_{35}.$

According to Eqs. (7), the mass transport velocity for the fifth-order Stokes wave is given by

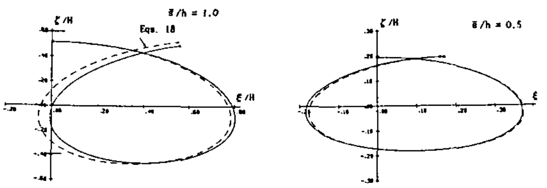
$$\frac{\bar{u}_L}{C_o}=(1+\varepsilon^2C_1'+\varepsilon^4C_2')\tanh kh\Big\{\frac{1}{32}[F_1^2(3F_1^2-F_2)$$
$$-F_1(5F_1^3+10F_1F_2-32F_{13}-16F_1)\cosh 2k\bar{s}$$
$$+(4F_1^4+31F_1^2F_2+16F_2^2)\cosh 4k\bar{s}]$$
$$-(S_{11}'\cosh k\bar{s}+S_{13}'\cosh 3k\bar{s}+S_{15}'\cosh 5k\bar{s})\cos k\bar{x}$$
$$-\frac{1}{2}(S_{20}'+S_{22}'\cosh 2k\bar{s}+S_{24}'\cosh 4k\bar{s})\cos 2k\bar{x}$$
$$-\frac{1}{3}(S_{31}'\cosh k\bar{s}+S_{33}'\cosh 3k\bar{s}+S_{35}'\cosh 5k\bar{s})\cos 3k\bar{x}$$
$$+\pi\frac{F_1^5}{32}(2\cosh k\bar{s}+5\cosh 3k\bar{s}+\cosh 5k\bar{s})\sin k\bar{x})\Big\} \quad (30a)$$

and

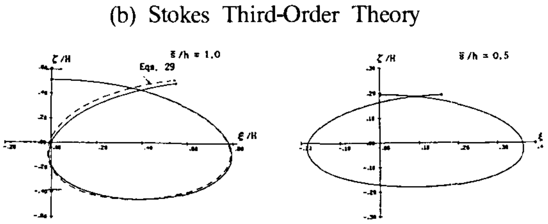
$$\frac{\bar{v}_L}{C_o}=(1+\varepsilon^2C_1'+\varepsilon^4C_2')\tanh kh\{-(T_{11}'\sinh k\bar{s}$$



(a) Stokes Second-Order Theory

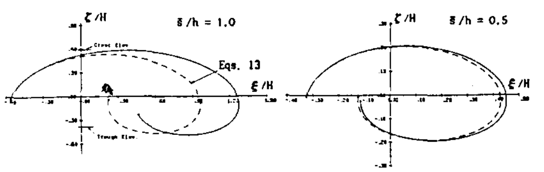


(b) Stokes Third-Order Theory

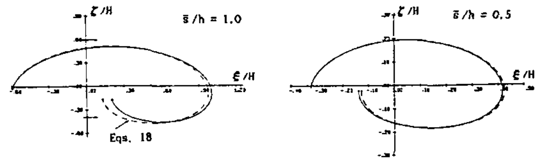


(c) Stokes Fifth-Order Theory

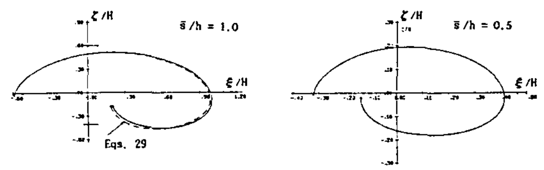
Fig. 2. Comparison between the predicted and the computed orbital motions of a water particle about its mean position at $k\bar{x}=0$; $H/L_o=0.0625$, $h/L_o=0.20$ [case7-B in Dean (1974)]



(a) Stokes Second-Order Theory



(b) Stokes Third-Order Theory



(c) Stokes Fifth-Order Theory

Fig. 3. Comparison between the predicted and the computed orbital motions of a water particle about its mean position at $k\bar{x}=\pi/4$; $H/L_o=0.0625$, $h/L_o=0.20$ [case7-B in Dean (1974)]

$$+T_{13}'\sinh 3k\bar{s}+T_{15}'\sinh 5k\bar{s})\sin k\bar{x}$$

$$-\frac{1}{2}T_{24}'\sinh 4k\bar{s}\cdot\sin 2k\bar{x}$$

$$-\frac{1}{3}(T_{31}'\sinh k\bar{s}+T_{33}'\sinh 3k\bar{s}+T_{35}'\sinh 5k\bar{s})\sin 3k\bar{x}$$

$$+\pi\frac{F_1^5}{16}(3\sinh k\bar{s}-\sinh 3k\bar{s})\cos k\bar{x})\} \qquad (30b)$$

The terms dependent on $\bar{x}$ in Eqs. (30a) tend to average to zero as a water particle travels over a distance equal to one wave-length, so that the long-term mass transport velocity reduces to

$$\frac{\bar{u}_L}{C_o}=(1+\varepsilon^2C_1'+\varepsilon^4C_2')\tanh kh\left\{\frac{1}{32}[F_1^2(3F_1^2-F_2)\right.$$
$$-F_1(5F_1^3+10F_1F_2-32F_{13}-16F_1)\cosh 2k\bar{s}$$
$$\left.+(4F_1^4+31F_1^2F_2+16F_2^2)\cosh 4k\bar{s}]\right\} \qquad (31)$$

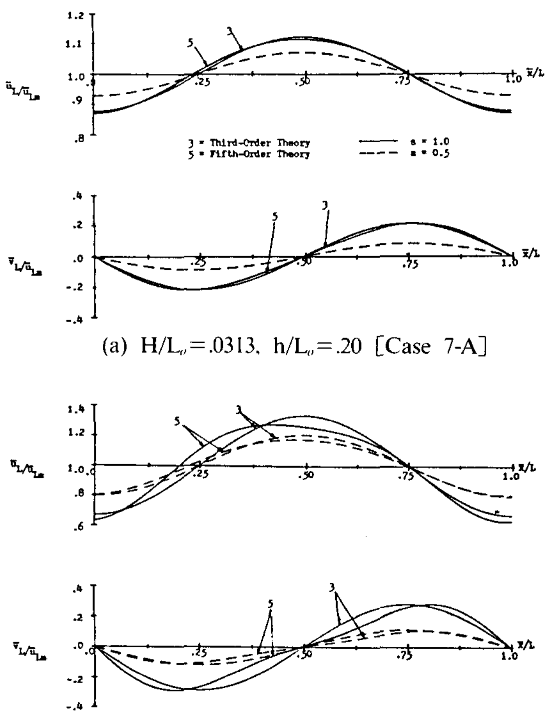
In deep water, Eq. (31) reduces to

$$\frac{\bar{u}_L}{C_o}=[1+(ak)^2+\frac{5}{4}(ak)^4]^{1/2}\cdot$$
$$[(ak)^2e^{2k\bar{y}}-\frac{5}{4}(ak)^4e^{2k\bar{y}}+(ak)^4e^{4k\bar{y}}] \qquad (32)$$

4. ANALYTICAL RESULTS

The orbital motion of the fluid particles whose mean position is located under the wave crest ($k\bar{x}=0$) at time $t=0$ is shown in Fig. 2 at the mean depths $\bar{s}/h=1.0$ (near the free surface) and $\bar{s}/h=.50$ (mid-depth) for waves with $h/L_o=.025$ and $h/L_o=.20$ [Case 7-B in Dean (1974)]. The solid lines represent the orbital motion computed by a numerical intergration of the water particle velocities given by the Stokes wave theory utilizing the fourth-order Runge-Kutta method of step-wise integration. The computer was fed the initial position of the water particle from which the computer calculated the local velocities and instantaneous direction of motion. After an increment of time ($t=.0125T$), the computer used the new particle position and repeated the necessary calculations. Reducing the time increment smaller than .0125T did not change the computed orbital motion. The dotted lines represent the orbital motions predicted by the expressions for water particle displacements ξ and ζ obtained in this study.

In Fig. 2, Stokes second-order theory is shown



(a) $H/L_o=.0313$, $h/L_o=.20$ [Case 7-A]

(b) $H/L_o=.0938$, $h/L_o=.20$ [Case 7-C]

Fig. 4. Variation of mass transport velocity about its mean value over one wave length.

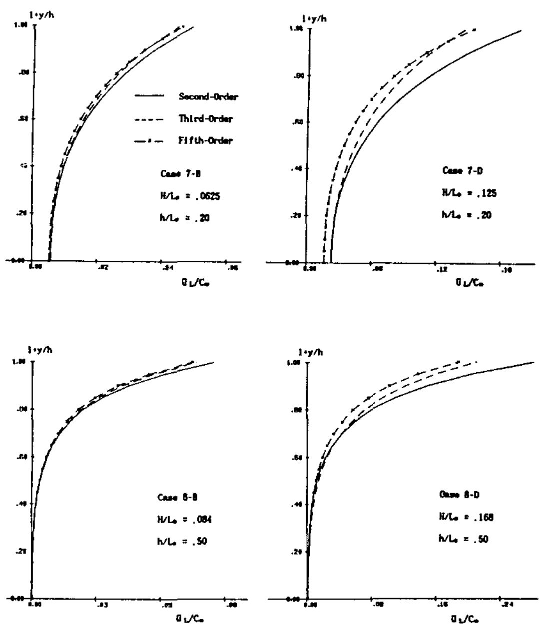


Fig. 5. Lagrangian mass transport velocity profiles over depth for Stokes waves.

to predict a larger horizontal mass transport velocity than higher-order theories especially near the free surface. The orbital motion predicted by Eqs. (13) has the horizontal mass transport velocity much greater than the computed values for second-order theory. For third- and fifth-order theory, the predicted orbital motion are close to the computed motions. The agreement between the predicted and computed motions according to the fifth-order Stokes wave theory at mid-depth is remarkable. The disagreement near the free-surface is due to the large displacement of the water particles from their mean positions which yields a poor approximation by Taylor's theorem. In Fig. 2, no vertical mass transport is found at $k\bar{x}=0$ as was predicted in the theories.

Fig. 3 shows the same comparisons for water particles whose mean position is located at a quarter wavelength from the crest ($k\bar{x}=\pi/4$) for the waves considered in Fig. 2. A large vertical mass transport in computed motions is noticeable as is in good agreement with the predicted values by the third- and the fifth-order theory. The third-and fifth-order expressions for the orbital motions given by Eqs. (18) and (29) slightly underpredict the horizontal mass transport in Fig. 2 and 3.

Computed motions at two different horizontal locations for the same wave condition show that the mass transport velocity varies according to the initial horizontal position of the water particles at time $t=0$ relative to the wave crest. This dependency of mass transport on the horizontal distance is not accounted for by the second-order theory, but is accounted for in Eqs. (19) and (30) for the third- and the fifth-order theories by the terms dependent on x.

Fig. 4 shows the variation of mass transport over one wave length according to the predictions by Eqs. (19) and (30) for the wave of (a) $H/L_o=0.031$, $h/L_o=0.20$ [Case 7-A in Dean (1974)], and (b) $H/L_o=0.094$, $h/L_o=0.20$ [Case 7-C in Dean (1974)]. The vertical axes represent the horizontal and vertical mass transport velocity, $\bar{u}_L$ and $\bar{v}_L$, to the mean horizontal mass transport velocity, u_{Lm}, given by the first term in Eqs. (19) and (30), respectively. The solid lines represent the values at the free surface ($\bar{s}/h=1.0$) and the dotted lines at the mid-depth (s/h=0.5). The maximum horizontal mass transport occurs with the particles whose mean positions are located under the wave trough at time t=0, while the minimum horizontal mass transport occurs with the particles whose mean positions are located under the wave crest at t=0. However, the opposite occurs in the case of the vertical mass transport. Variations of mass transport velocity about its mean values are most pronounced at the free surface and decreases with increasing vertical distance from the free surface. The ratio of $\bar{u}_L/\bar{u}_{Lm}$ varies between $-0.6<\bar{u}_L/\bar{u}_{Lm}<1.3$ and $-0.3<\bar{v}_L/\bar{u}_{Lm}<0.3$ at the free-surface over the range of $0.08<h/L_o<2.0$ and $0.02<H/L_o<0.12$.

As water particles travel over a large distance longer than one wave length, the mass transport due to the mass transport components which are dependent on $\bar{x}$ in Eqs. (19) and (30) tend to average to zero, and the average horizontal mass transport velocity may be given by the first terms in Eqs. (19a) and (30a). The average (or long-term) horizontal mass transport velocity profiles over depth are plotted in Fig. 5 for the four wave conditions.

In Fig. 5, Case 7 and Case 8 by Dean's (1974) category represent the intermediate water depth and deep water, respectively, while condition B and D represent wave height of 50% and 100% of the breaking limit, respectively. In general, the higher-order Stokes theories predict smaller values of mass transport velocity than the existing second-order Stokes theory over the whole water depth, and the difference is increasing as the wave height approaches to the breaking condition. At the free-surface, the mass transport (surface drift) velocities in the 3rd-order and 5th-order theory show 88-93% of those in the second-order theory for the cases of 7-B and 8-B. For the cases of 7-D and 8-D, however, the surface drift velocities in the 3rd- and 5th-order theory predict 57-79% of those in the second-order theory. This results show that the existing second-order Stokes mass transport velocity should be modified significantly, especially near the free-surface for waves of large amplitude.

At present, experimental data which provide definite comparison between theories for the mass transport velocity of water particles under progressive waves are not available. Therefore, judgment concerning the excellency of each mass transport theory presented in this study may be made when experimental data which are acquired through more highly controlled laboratory measurement than the previous experimental studies are reported.

5. CONCLUSIONS

The existing theories for orbital motion and mass transport velocity under progressive finite-amplitude waves are extended to include higher-hamonic wave components by utilizing Taylor's theorem. The 5th-order approximation of orbital motion for the Stokes waves gives very good predictions of actual water particle motion given by the Stokes 5th-order theory except near the free surface. Near the free surface, the 5th-order approximation for the Lagrangian motion of water particles predicts slightly lower values of mass transport velocity than the actual values computed by the Stokes 5th-order wave theory.

In the 3rd- or higher-order approximation for the Lagrangian motion of water particles, the mass transport velocity over one wave cycle is dependent on the initial position of water particles. The variation of the horizontal mass transport velocity about its mean value over one wave cycle at the free surface is 40% at most for the wave steepness parameter $0.02<H/L_o<0.12$. Stokes 3rd- and 5th-order theories predict the mass transport velocity less than that given by the existing second-order theory over the whole water depth, and the difference increases as the wave amplitude increases. At the breaking limit, the surface drift velocities in the 3rd-and 5th-order theory show approximately 70-80% of the values in the second-order theory in intermediate and deep water conditions.

REFERENCES

Bijker, E.W., Kalkwijk, J.P. Th. and Pieters, T., 1974. Mass transport in gravity waves on a sloping bottom, Proceedings of the 14th Conference on Coastal Engineering, ASCE, Copenhagen, Denmark, June 24-28, pp. 447-465

Caligny, A.F.H. de C.R., 1978. Expériences sur les mouvements des molécules liquides des ondes courantes, considérées dans leur mode d'action sur la marche des navires, C.R. Acad. de Sci., Paris, **87**: 1019-1023.

Craik, A.D.D., 1982. The drift velocity of water waves, *Journal of Fluid Mechanics*, **116**: 187-205.

Dalrymple, R.A., 1976. Wave-induced mass transport in water waves, Journal of Waterways, Harbors and Coastal Engineering Division, ASCE, **102**(WW2): 255-264.

Dean, R.G., 1974. Evaluation and development of water wave theories for engineering application, Volume 1. Special Report No. 1, U.S. Army Corps of Engineers, Coastal Engineering Research Center. Fort Belvoir, Virginia. U.S.A.

Kim, T-I. and Hudspeth, R.T., 1988. Mass transport in finite amplitude waves, Proceedings of the 6th APD Congress, IAHR, Kyoto, Japan, III: 17-24.

Mei, C.C., 1983. The applied dynamics of ocean surface waves, John Wiley & Sons, Inc., New York, New York. U.S.A., Chapters 8 and 9.

Mei, C.C., Liu, P. L-F. and Carter, T.G., 1972. Mass transport in water waves, Report 146, Ralph M. Parsons Laboratory for Water Resources and Hydrodynamics, Massachusetts Institute of Technology, Cambridge, Massachusetts, U.S.A.

Nishimura, H., Isobe, M. and Horikawa, K., 1977. Higher-order solutions of the Stokes and the Cnoidal waves, Jounal of the Faculty of Engineering, U. of Tokyo, **34**(92): 267-293.

Rayleigh, L., 1986. On waves. *Philosophical Magazine*, London, Dublin & Edinburgh, U.K., **5**(1), pp. 257-279.

Russell, R.C.H. and Osorio, J.D.G., 1958. An experimental investigation of drift profiles in a closed channel, Proceedings of the 6th Conference on Coastal Engineering, Council on Wave Research, ASCE, U. of California, Berkeley, California, U.S.A., 171-193.

Skjelbreia, L., 1959. Gravity Waves: Stokes Third-order Appoximation, Tables of Functions, The Engineering Foundation Council on Wave Research, Berkeley, CA., U.S.A., pp. 231-251.

Skjelbreia, L. and Hendrickson, J.H., 1960. Fifth-order gravity wave theory, Proceedings of the 7th Conference on Coastal Engineering, Hague, The Netherlands, pp. 184-196.

Stokes, G.G., 1847. On the theory of oscillatory waves, *Transactions of the Cambridge Philosophical Society*, London, England, **8**: 441-455.

Tsuchiya, Y., Yasuda T. and Yamashita, T., 1980. Mass transport in progressive waves of permanent type, Proceedines of the 17th Conference on Coastal Engineering, ASCE, Sidney, Australia, March, pp. 70-81.

Ursell, F., 1953. Mass transport in gravity waves, *Proceedings of the Cambridge Philosophical Society*, London, England, October, **49**: 145-150.

U.S. Beach Erosion Board, 1941. A study of progressive oscillatory waves in water, Technical Report No. 1, Government Printing Office, Washington D.C., U.S.A.

Wang, H., Sunamura, T. and Huang, P.A., 1982. Drift velocity of the wave breaking point, Coastal Engineering, The Netherlands, **6**: 121-150.

Wehausen, J.V., 1960. Surface waves in Handbuch der physik, Volume 9, S. Flügge editor, Springer-Verlag, Berlin, 446-757.

Wiegel, R.L., 1964. Oceanographical Engineering, Prentice Hall, Inc., Englewood Cliffs, N.Y., U.S.A., pp. 28-40.

APPENDIX I.
COEFFICIENTS USED IN STOKES FIFTH-ORDER WAVE THEORY

The coefficients used in the Stokes fifth-order wave theory were obtained by Skjelreia and Hendrickson (1960) as follows:

$$A_{11}=\frac{1}{s}, \qquad A_{13}=\frac{-c^2(5c^2+1)}{8s^5}$$

$$A_{15}=\frac{-(1184c^{10}-1440c^8-1992c^6+2641c^4-249c^2+18)}{1536s^{11}}$$

$$A_{22}=\frac{3}{8s^4}, \qquad A_{24}=\frac{(192c^8-424c^6-312c^4+480c^2-17)}{768s^{10}}$$

$$A_{33}=\frac{(13-4c^2)}{64s^7}$$

$$A_{35}=\frac{(512c^{12}+4224c^{10}-6800c^8-12808c^6+16704c^4-3154c^2+107)}{4096s^{13}(6c^2-1)}$$

$$A_{44}=\frac{(80c^6-816c^4+1338c^2-197)}{1536s^{10}(6c^2-1)}$$

$$A_{55}=\frac{-(2800c^{10}-72480c^{8}+324000c^{6}-432000c^{4}+163470c^{2}-16245)}{61440s^{11}(6c^{2}-2)\ (8c^{4}-11c^{2}+3)}$$

$$B_{33}=\frac{3(8c^{6}+1)}{64s^{6}}$$

$$B_{35}=\frac{(88128c^{14}-208224c^{12}+70848c^{10}+54000c^{8}-21816c^{6}+6264c^{4}-54c^{2}-81)}{12288s^{12}(6c^{2}-1)}$$

$$B_{55}=\frac{(192000c^{16}-262720c^{14}+83680c^{12}+20160c^{10}-7280c^{8})}{12228s^{10}(6c^{2}-1)\ (8c^{4}-11c^{2}+3)}+\frac{(7160c^{6}-1800c^{4}-1050c^{2}+225)}{12288s^{10}(6c^{2}-1)\ (8c^{4}-11c^{2}+3)}$$

$$C_1'=\frac{(8c^{4}-8c^{2}+9)}{8s^{4}}$$

$$C_2'=\frac{(3840c^{12}-4096c^{10}-2592c^{8}-1008c^{6}+5944c^{4}-1830c^{2}+147)}{512s^{10}(6c^{2}-1)}$$

in which $c=\cosh kh$, and $s=\sinh kh$.