Structures of Fuzzy Relations

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Abstract

In this paper we consider the notion of fuzzy relation as a generalization of that of fuzzy set. For a complete Heyting algebra L, the category *Set*(L) of all L-fuzzy sets is shown to be a bireflective subcategory of the category *Rel*(L) of all L-fuzzy relations and L-fuzzy relation preserving maps. We investigate categorical structures of subcategories of *Rel*(L) in view of quasitopos. Among those categories, we include the category L-fuzzy similarity relations with respect to both max-min and max-product compositions, respectively, as a cartesian closed topological category. Moreover, we describe exponential objects explicitly in terms of function space.

1. Introduction

Fuzzy relations are essential tools in many applications of fuzzy sets. Therefore, the study of fuzzy relations has its own value in both mathematical and applicational point of view. Over the last 25 years, many researchers have studied structure of fuzzy sets in a categorical point of view to expose the relationship between fuzzy mathematics and topoi with a view to make a true "fuzzy logic" both consistent, complete and respectable [2-8,12,13].

In this paper we consider the notion of fuzzy relation as a generalization of that of fuzzy set and study structures of fuzzy relations in a categorical viewpoint. For a complete Heyting algebra L, we denote the category of all L-fuzzy sets defined by Goguen [6] by *Set*(L). It is shown that *Set*(L) is a bireflective subcategory of the category *Rel*(L) of all L-fuzzy relations and L-fuzzy relation preserving maps. We investigate subcategories of *Rel*(L) in view of quasitopos [9]. Among those categories, we include L-fuzzy similarity relations with respect to both max-min and max-product compositions. Moreover, we describe exponential objects explicitly in terms of function space. For categorical background, we refer to [1].

2. Fuzzy sets and fuzzy relations

Let L be a complete Heyting algebra. A category *Set*(L) [6] is defined by L-fuzzy sets (X,μ) and

*This research was supported by KOSEF grant 91-0100-20

functions $f:(X,\mu)\rightarrow(Y,\nu)$ such that $\mu \le \nu \circ f$. In a categorical point of view, this category *Set*(L) has long been studied and justified in many ways [2-8,12,13]. In a similar way now we form a category ***Rel***(L) as follows: objects are the pair (X,R), where X is a set and R is a function from $X \times X$ into L, i.e. an L-fuzzy relation on X and morphisms are functions $f:(X,R)\rightarrow(Y,S)$ such that $R \le S \circ f$. In this case f is called an L-fuzzy relation preserving map.

Given an L-fuzzy set (X,μ), we define an L-fuzzy relation $R_\mu(x,x') = \mu(x) \wedge \mu(x')$. Let E be a functor from *Set*(L) into ***Rel***(L) defined by $E(X,\mu) = (X,R_\mu)$ and $E(f) = f$. Then it is easy to that check E is a full embedding functor. Hence we may consider *Set*(L) as a subcategory of ***Rel***(L). Moreover, we have the following result.

Theorem 2.1. *Set*(L) is a bireflective subcategory of ***Rel***(L).

Proof. Given an L-fuzzy relation (X,R), we have an L-fuzzy set $\mu_R(x) = (\bigvee_{y\in X} \mu(x,y)) \vee (\bigvee_{y\in X} \mu(y,x))$ on X. In fact, $(id_X, (X,\mu_R))$ is a reflection of (X,R).

3. The category *Rel*(L)

We recall some results on basic structures of L-fuzzy relations from [10]. Consider the underlying functor $U:$ ***Rel***(L) $\rightarrow$ *Set* defined by $U(X,R) = X$ and $U(f) = f$.

Theorem 3.1. [10] The functor U is topological over *Set*.

Proof. Let X be a set and $\{f_i : X \rightarrow (X_i,R_i)\}_I$ a family of functions. Then $R(x,y) = \bigwedge_{i\in I} R_i(f_i(x),f_i(y))$ give an initial L-fuzzy relation on X with respect to $\{f_i\}_I$.

Remark. Given a family $\{f_i:(X_i,R_i)\rightarrow X\}_I$, the final L-fuzzy relation R on X is obtained from the formula $R(x,y) = \bigvee_{i\in I}\,\bigvee_{(a,b)\in(f_i\times f_i)^{-1}(x,y)} R_i(a,b)$.

Theorem 3.2. [10] In ***Rel***(L), final epi-sinks are preserved by pull backs.

Remark. We note that the category ***Rel***(L) is not a topos, since a bimorphism is not necessarily an isomorphism. Even though a singleton have more than one structure and hence constant map is not necessarily a morphism, it is shown [10] that ***Rel***(L) is cartesian closed by a modification of [2]: For $(X,R),(Y,S) \in$ ***Rel***(L), let $C(X,Y)$ be the set of morphisms from (X,R) into (Y,S). Then $T(f,g) = \vee\{\alpha \in L \mid R(x,x') \wedge \alpha \le S(f(x),g(x'))$ for all $x,x' \in X\}$ gives an L-fuzzy relation on $C(X,Y)$ allowing ***Rel***(L) to be cartesian closed.

Theorem 3.3. [10] ***Rel***(L) is cartesian closed.

4. Subcategories of *Rel*(L)

An L-fuzzy relation R on X is reflexive if $R(x,x)=1$ for all $x \in X$. Let ***Rel***$_R$(L) be the subcategory of ***Rel***(L) formed by all reflexive L-fuzzy relations. By the formulas of initial and final structures in ***Rel***(L), it is easy to see that ***Rel***$_R$(L) is closed under the formation of initial sources and final-epi-sinks in ***Rel***(L), respectively. Hence ***Rel***$_R$(L) is a bi(co)reflective subcategory of ***Rel***(L). In fact, given $(X,R) \in$ ***Rel***(L), let R^* be the function defined by $R^*(x,y)=R(x,y)$ if $x \ne y$, and $R^*(x,x)=1$ for all $x \in X$. Then $(id,(X,R^*))$ is a reflection of (X,R). Moreover, initial and final structures in ***Rel***$_R$(L) can be obtained from ***Rel***(L) using reflections. We note that every singleton has a unique structure in ***Rel***$_R$(L) and hence every constant map is a morphism.

By some modification of the proof of Theorem 3. 2. , it is shown that

Theorem 4.1. [10] In $Rel_R(L)$, final epi-sinks are preserved by pull-backs and hence $Rel_R(L)$ is a quasitopos.

Remark. As a corollary, the category $Rel_R(L)$ is cartesian closed (cf. [11]). However the cartesian closedness of $Rel_R(L)$ is shown directly by the L-fuzzy relation on a function space C(X,Y) in remark 2. 2. Moreover, in the case of a chain L, we can obtain an internal description of the L-fuzzy relation T on C(X,Y) as follows: For $f,g \in C(X,Y)$, let $D(f,g) = \{(x,x') \mid R(x,x') > S(f(x),f(x'))\}$. Then $T(f,g) = 1$, if $D(f,g) = \phi$, and $T(f,g) = \bigwedge_{(x,x') \in D(f,g)} S(f(x),g(x'))$, otherwise (cf. [14]).

Theorem 4.2. $Rel_T(L)$ and $Rel_t(L)$ are closed under the formation of initial sources in $Rel(L)$, respectively.

Proof. Let $\{f_i : (X,R) \to (X_i,R_i)\}_I$ be an initial source in $Rel(L)$. Suppose each R_i is transitive with respect to max-min composition. Then

$$
\begin{aligned}
R \circ R(x,y) &= \vee_z [\wedge_i R_i(f_i(x),f_i(z))] \wedge [\wedge_i R_i(f_i(z),f_i(y))] \\
&\leq \vee_z R_i(f_i(x),f_i(z)) \wedge R_i(f_i(z),f_i(y)) \\
&\leq R_i \circ R_i (f_i(x),f_i(y)) \\
&\leq R_i(f_i(x),f_i(y))
\end{aligned}
$$

for each $i \in I$. Hence $R \circ R(x,y) \leq \wedge_i R_i(f_i(x),f_i(y)) = R(x,y)$. Therefore R is transitive. By a similar argument, this result holds for max-product composition.

Corollary 4.3. $Rel_T(L)$ and $Rel_t(L)$ are bireflective subcategories of $Rel(L)$.

Remark. Since $Rel_T(L)$ and $Rel_t(L)$ have initial structures, they have final structures. However they are not closed under the formation of final epi-sinks in $Rel(L)$. Let $\{f_i : (X_i,R_i) \to X\}_I$ be a family of functions and R the final structure with respect to $\{f_i\}$ in $Rel(L)$. Then the transitive closure R^* of R is the final structure of for the family $\{f_i\}$ in $Rel_T(L)$ and $Rel_t(L)$, respectively.

Theorem 4.4. $Rel_T(L)$ is a bireflective subcategory of $Rel_t(L)$.

Proof. Since $R \cdot R \leq R \circ R$, $Rel_T(L) \subseteq Rel_t(L)$. ($R \cdot R$ means the max-product composition.) By Theorem 4.2., the result follows.

Let $Rel_{PM}(L)$ ($Rel_{PP}(L)$, resp.) be the subcategory of $Rel(L)$ formed by all L-fuzzy preorder relations with respect to max-min (max-product, resp.) compositions. By the above results, the categories $Rel_{PM}(L)$ and $Rel_{PP}(L)$ are topological and they are bireflective subcategories of $Rel(L)$.

Theorem 4.5. $Rel_{PM}(L)$ and $Rel_{PP}(L)$ are cartesian closed.

Proof. For $(X,R), (Y,S) \in Rel(L)$, let C(X,Y) be the set of morphisms from (X,R) into (Y,S). Let $T(f,g) = \vee \{\alpha \in L \mid R(x,x') \wedge \alpha \leq S(f(x),g(x'))$ for all $x,x' \in X\}$. Then clearly T is reflexive.

$$
\begin{aligned}
T \circ T(f,g) &= \vee_h T(f,h) \wedge T(h,g) \\
&= \vee_h [(\vee \{\alpha \in L \mid R(x,x') \wedge \alpha \leq S(f(x),h(x')) \text{ for all } x,x' \in X\}) \wedge \\
&\qquad (\vee \{\beta \in L \mid R(y,y') \wedge \beta \leq S(h(y),g(y')) \text{ for all } y,y' \in X\})] \\
&\leq \vee_h \vee \{\gamma \in L \mid R(x,x') \wedge R(y,y') \wedge \gamma \leq S(f(x),h(x')) \wedge \\
&\qquad S(h(y),g(y')) \text{ for all } x,x' \in X,\ y,y' \in X\} \\
&\leq \vee_h \vee \{\gamma \in L \mid R(x,z) \wedge R(z,y) \wedge \gamma \leq S(f(x),h(z)) \wedge
\end{aligned}
$$

$S(f(z),g(y))$ for all $x,y,z \in X\}$
$\leq \vee\{\gamma \in L \mid R \circ R(x,y) \wedge \gamma \leq S \circ S(f(x),g(y))$ for all $x,y \in X\}$
$= \vee\{\gamma \in L \mid R(x,y) \wedge \gamma \leq S(f(x),g(y))$ for all $x,y \in X\}$
$= T(f,g)$

Hence T is a L-fuzzy preorder relation on C(X,Y). Therefore by the Remark of Theorem 3.2., it is easy to check that $\boldsymbol{Rel}_{PM}$(L) is cartesian closed. Moreover by a similar argument we can show that $\boldsymbol{Rel}_{PP}$(L) is cartesian closed.

Let $\boldsymbol{Rel}_{SM}$(L) ($\boldsymbol{Rel}_{SP}$(L), resp.) be the subcategory of $\boldsymbol{Rel}$(L) formed by all L-similarity relations with respect to max-min (max-product, resp.) compositions. By the above arguments, it is easy to see that $\boldsymbol{Rel}_{SM}$(L) and $\boldsymbol{Rel}_{SP}$(L) are topological and they are bireflective subcategories of $\boldsymbol{Rel}$(L). Moreover using the proof of Theorem 4.5., we obtain the following.

Theorem 4.6. $\boldsymbol{Rel}_{SM}$(L) and $\boldsymbol{Rel}_{SP}$(L) are cartesian closed.

Remarks. In the case of a chain L, by a natural modification of the proof in [14], we can show that the L-fuzzy relation T on C(X,Y) has an internal description, as in Remark of Theorem 4. 2., for both $\boldsymbol{Rel}_{SM}$(L) and $\boldsymbol{Rel}_{SP}$(L).

The categories mentioned above are not topoi, because a bimorphism needs not be an isomorphism (cf. [10]). However a question arises whether $\boldsymbol{Rel}_{PM}$(L), $\boldsymbol{Rel}_{PP}$(L), $\boldsymbol{Rel}_{SM}$(L), and $\boldsymbol{Rel}_{SP}$(L) are quasitopoi. In a separate paper we will discuss about this matter and present some results on structures of L-fuzzy preorder relations, L-similarity relations, L-fuzzy order relations, L-fuzzy perfect order relations and L-fuzzy total order relations, etc. with respect to s-norms and t-norms.

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