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Extended Quasi-likelihood Estimation in Overdispersed Models†

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ABSTRACT

Samples are often found to be too heterogeneous to be explained by a one-parameter family of models in the sense that the implicit mean-variance relationship in such a family is violated by the data. This phenomenon is often called over-dispersion. The most frequently used method in dealing with over-dispersion is to mix a one-parameter family creating a two parameter marginal mixture family for the data. In this paper, we investigate performance of estimators such as maximum likelihood estiamtor, method of moment estimator, and maximum quasi-likelihood estimator in negative binomial and beta-binomial distribution. Simulations are done for various mean parameter and dispersion parameter in both distributions, and we conclude that the moment estimators are very superior in the sense of bias and asymptotic relative efficiency.

KEYWORDS: Beta-binomial distribution, Negative binomial distribution, Over-dispersion, Quasi-likelihood.

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1. INTRODUCTION

Analysis of data via a single parameter of family of distributions implies in particular that the variance is determined by the mean. Familiar examples are the Poisson, binomial and exponential distributions. However, samples are often found to be too heterogeneous to be explained by a one-parameter family of models in the sense that the implicit mean-variance relationship in such a family is violated by the data; the sample variance is large compared with that predicted by inserting the sample mean into the mean-variance relationship. This phenomenon is often called over-dispersion. The most frequently used method in dealing with over-dispersion is to mix a one-parameter family creating a two-parameter marginal mixture family for the data. Morris (1982,1983) defined natural exponential family (NEF) by mixing one-parameter exponential family with two-parameter conjugate mixture (CM) distribution. Cox (1983) noted that for modest amounts of over-dispersion a full specification of the mixing distribution was unnecessary, but its mean and variance are needed. To model over-dispersion, Efron (1986) creates so called double exponential family, and Lindsay (1986) addresses the slightly different question of whether mixing a one-parameter exponential family can produce a two-parameter exponential family. Jorgensen (1987) extended the one-parameter exponential family to a two-parameter class of distributions so called an exponential dispersion model. Also, Gelfand and Dalal (1990) suggested two-parameter exponential family containing Lindsay's as a special case, and recommend to use weighted least squares estimate for the over-dispersion parameter.

In this paper, we adopt negative-binomial distribution and beta-binom-ial distribution as models for over-dispersed Poisson and binomial data, respectively, and we study the asymptotic relative efficiencies (ARE) of the maximum quasi-likelihood (McCullagh and Nelder, 1983; Nelder and Pregibon, 1987; McCullagh and Nelder, 1990) estimator (QL) and the method of moment estimator (MM) with respect to the maximum likelihood estimator (ML) of mean parameter μ and dispersion parameter ϕ. Kleinman (1973) studied the ARE of the weighted moment estimator with respect to least squares estimator for over-dispersed binomial data. The ARE of QL with respect to ML of unknown parameter β in over-dispersed model has been studied by Firth (1987), and the ARE of QL with respect to ML of μ in negative binomial and beta-binomial was discussed by Hill and Tsai (1988) under the assumption of the dispersion parameter ϕ being known.

Section 2 describes Fisher information matrices of (μ, ϕ) for the ML, QL, and MM extending the results of Hill and Tsai (1988). The AREs of QL and MM with respect to ML are numerically obtained for various μ's and ϕ's in Section 3.

2. ESTIMATION OF PARAMETERS AND ASYMPTOTIC VARIANCES

We assume that, conditional on the sampling means θ_i, the data y_i have independent distributions belonging to a natural exponential family (NEF) with quadratic variance function (Morris:1982,1983), and that the means θ_i are independent with conjugate mixture (CM) distributions. Let $[\mu, V(\mu)]$ denote a distribution with mean μ and variance function $V(\mu)$.

2.1 Gamma-Poisson mixtures

The data for this example, given in Table 1, is Accident data (Seal, 1969) consisting of observed counts of accidents in a year for 9461 Belgian drivers. By taking on initial Poisson distribution, nominal variance is $\bar{y} = .214$ with sample variance $s^2 = .289$ suggesting over-dispersion. These data have been analyzed by Lindsay (1986) and Gelfand and Dalal (1990). To define the model for these data,

$$y|\theta \sim \text{Poisson}(\theta) = \text{NEF}[\theta, \theta],$$

$$\theta \sim \Gamma\left(\frac{\mu}{\phi}, \phi\right) = \text{CM}[\mu, \phi\mu],$$

$$V(\mu) = \mu$$

and

$$y \sim \text{NB}\left(\frac{\mu}{\phi}, \frac{\phi}{1+\phi}\right) = \text{marg}[\mu, (1+\phi)\mu] \tag{2.1}$$

where $\Gamma(\alpha, \beta)$ denotes a gamma distribution with parameters α and β, and $\text{NB}(\alpha, \beta)$ denotes a negative binomial distribution with probability function

$$p(y) = \binom{\alpha + y - 1}{y} \beta^y (1-\beta)^\alpha, \quad y = 0, 1, \cdots$$

Table 1. Accident data

y	0	1	2	3	4	5	6	7
count	7840	1317	239	42	14	4	4	1

The marginal log-likelihood corresponding to (2.1) is

$$l(\mu,\phi)=\sum_{j=1}^{y}\log\left(\frac{\mu}{\phi}+j-1\right)-\frac{\mu}{\phi}\log(1+\phi)+y\log\left(\frac{\phi}{1+\phi}\right).$$

The maximum likelihood estimates $\hat{\mu}_{ML}$ and $\hat{\phi}_{ML}$ can be calculated by using the profile likelihood approach. For the expected Fisher information matrix, we find after a little effort that

$$l_{11}=E\left[-\frac{\partial^2 l}{\partial\mu^2}\right]=\sum_{y=0}^{\infty}p(y)\sum_{j=1}^{y}[\mu+(j-1)\phi]^{-2}$$

$$l_{12}=E\left[-\frac{\partial^2 l}{\partial\mu\partial\phi}\right]=\sum_{y=0}^{\infty}p(y)\sum_{j=1}^{y}(j-1)\,[\mu+(j-1)\phi]^{-2}-\frac{1}{\phi^2}\log(1+\phi)+\frac{1}{\phi(1+\phi)}$$

$$l_{22}=E\left[-\frac{\partial^2 l}{\partial\phi^2}\right]=-\frac{\mu}{\phi^2}\sum_{y=0}^{\infty}p(y)\sum_{j=1}^{y}\left\{\frac{\mu+2(j-1)\phi}{[\mu+(j-1)\phi]^2}\right\}+\frac{2\mu}{\phi^3}\log(1+\phi)-\frac{\mu}{\phi^2(1+\phi)}.$$

The extended quasi-likelihood function becomes

$$q_i(\mu,\phi)=\frac{1}{1+\phi}\{(y_i\log\mu-\mu)-(y_i\log y_i-y_i)\}-\frac{1}{2}\log(1+\phi)$$

and the maximum quasi-likelihood estimates $\hat{\mu}_{QL}$ and $\hat{\phi}_{QL}$ are given by

$$\hat{\mu}_{QL}=\bar{y},$$

$$\hat{\phi}_{QL}=\tfrac{1}{n}\sum d_i-1$$

where $\bar{y}$ is sample mean of y's and

$$d_i=-2\{(y_i\log\mu-\mu)-(y_i\log y_i-y_i)\}.$$

An adjusted version of $\hat{\phi}_{QL}$ can be derived by using

$$E(d)\simeq(1+\phi)(1+b) \tag{2.2}$$

where

$$b=(1+\phi)/6\mu \tag{2.3}$$

is a Bartlett adjustment. Let $\hat{\phi}_{AQL}$ be an estimator satisfying (2.2), i.e., $\hat{\phi}_{AQL}=-(3\hat{\mu}+1)+\{3\hat{\mu}(3\hat{\mu}+2\bar{d})\}^{1/2}$. Note that iterations are not needed in calculating $\hat{\mu}_{QL}$ and $\hat{\phi}_{QL}$. For the asymptotic variance of $\hat{\mu}_{QL}$, it is straightforward that

$$n\,Var(\hat{\mu}_{QL}) = (1+\phi)\mu.$$

As suggested by McCullagh and Nelder (1990), however, adjustment should be made for the variance of $\hat{\phi}_{QL}$ in such a way that

$$n\,Var(\hat{\phi}_{QL}) \simeq 2(1+\phi)^2(1+b)^2$$

where b is defined in (2.3).

The method of moment estimates $\hat{\mu}_{MM}$ and $\hat{\phi}_{MM}$ are given by equating

$$\mu = \bar{y},$$

$$(1+\phi)\mu = s^2$$

where s^2 denotes sample variance of y's. Therefore,

$$\hat{\mu}_{MM} = \bar{y},$$

$$\hat{\phi}_{MM} = \frac{s^2}{\bar{y}} - 1.$$

To get the asymptotic variances for the moment estimates, note that (Serfling, 1980, p.72)

$$\sqrt{n}\begin{pmatrix} \bar{y}-\mu_1 \\ s^2-\mu_2 \end{pmatrix} \xrightarrow{\mathcal{L}} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \mu_2 & \mu_3 \\ \mu_3 & \mu_4-{\mu_2}^2 \end{pmatrix}\right)$$

where

$$\mu_k = E\left[(Y-\mu)^k\right].$$

Explicit forms for μ_k are easily obtained by noting that (McCullagh and Nelder, 1990, p.350)

$$\kappa_{r+1} = {\kappa_r}'\kappa_2, \quad r \geq 2$$

where κ_r is the r-th cumulant. Hence, we have

$$\mu_2 = (1+\phi)\mu,$$

$$\mu_3 = (1+\phi)^2\mu,$$

$$\mu_4 = \kappa_4 + 3\kappa_2{}^2 = \mu(1+\phi)^3 + 3\mu^2(1+\phi)^2$$

and the asymptotic variances for $\hat{\mu}_{MM}$ and $\hat{\phi}_{MM}$ are easily obtained by the multivariate delta method, i.e.,

$$nVar(\hat{\mu}_{MM}) = (1+\phi)\mu,$$

$$nVar(\hat{\phi}_{MM}) = 2(1+\phi)^2.$$

Hence,

$$\text{ARE}(\hat{\mu}_{QL}, \hat{\mu}_{MM}) = 1,$$

$$\text{ARE}(\hat{\phi}_{QL}, \hat{\phi}_{MM}) = \left(1 + \tfrac{1+\phi}{6\mu}\right)^{-2}.$$

2.2 Beta-Binomial mixture

The data for the beta-binomial mixture could be Sibship data (Sokal and Rohlf, 1973) in Table 2, which has been analyzed by Gelfand and Dalal (1990), consist of the frequency of males in 6115(n) sibships of size 12(m) given the probability of success θ, the observed response rate y is assumed to follow binomial distribution, θ has a beta distribution, and let $r = my$ be observed frequency. Under a binomial distribution, nominal variance is $m\hat{\theta}(1-\hat{\theta}) = 2.9956$ with sample variance $s^2 = 3.4886$ suggesting over-dispersion. In symbols,

$$y|\theta \sim \frac{1}{m}\text{Binomial}(m,\theta) = \text{NEF}\left[\theta, \frac{\theta(1-\theta)}{m}\right]$$

$$\theta \sim \text{Beta}(\psi\mu, \psi(1-\mu)) = \text{CM}[\mu, \phi\mu(1-\mu)]$$

where $\psi = 1/\phi - 1$, and the variance function in this situation is $V(\mu) = \mu(1-\mu)$. Then, the marginal distribution of y becomes

$$y \sim \frac{1}{m}\text{BB}(m, \psi\mu, \psi(1-\mu)) = \text{marg}[\mu, \mu(1-\mu)/w]$$

where $\text{BB}(m,\alpha,\beta)$ denotes a beta-binomial random variable with probability function

$$p(r) = \binom{m}{r}\frac{\Gamma(\alpha+\beta)\Gamma(r+\alpha)\Gamma(m+\beta-r)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(m+\alpha+\beta)}, \quad r = 0, 1, \ldots, m$$

and $w = m/\{1+(m-1)\phi\}$. Hence, $\phi > 0$ implies over-dispersion.

Table 2. Sibship data

r	0	1	2	3	4	5	6	7	8	9	10	11	12
count	3	24	104	286	670	1033	1343	1112	829	478	181	45	7

The marginal log-likelihood is given by

$$l(\mu,\psi)=\sum_{j=1}^{r}\log(\psi\mu+j-1)+\sum_{j=1}^{m-r}\log\{\psi(1-\mu)+j-1\}-\sum_{j=1}^{m}\log(\psi+j-1).$$

The ML's $\hat{\mu}_{ML}$ and $\hat{\phi}_{ML}$ are obtained easily by the functional invariance property and similar method used in Section 2.1. The expected Fisher information matrix for (μ,ϕ) can be calculated by using

$$\frac{\partial^2 l}{\partial\mu\partial\phi}=\frac{\partial^2 l}{\partial\mu\partial\psi}\frac{\partial\psi}{\partial\phi}$$

and

$$\frac{\partial^2 l}{\partial\phi^2}=\left.\frac{\partial^2 l}{\partial\psi^2}\right]_{\psi=\frac{1}{\phi}-1}\left(\frac{\partial\psi}{\partial\phi}\right)^2$$

then, we have

$$E\left[-\frac{\partial^2 l}{\partial\mu^2}\right]=\psi^2\left[\sum_{r=0}^{m}p(r)\left\{\sum_{j=1}^{r}(\psi\mu+j-1)^{-2}-\sum_{j=1}^{m-r}(\psi(1-\mu)+j-1)^{-2}\right\}\right]$$

$$E\left[-\frac{\partial^2 l}{\partial\mu\partial\phi}\right]=\frac{1}{\phi^2}\left[\sum_{r=0}^{m}p(r)\left\{\sum_{j=1}^{r}\frac{j-1}{(\psi\mu+j-1)^2}-\sum_{j=1}^{m-r}\frac{j-1}{(\psi(1-\mu)+j-1)^2}\right\}\right]$$

$$E\left[-\frac{\partial^2 l}{\partial\phi^2}\right]=\frac{1}{\phi^4}\left[\sum_{r=0}^{m}p(r)\left\{\sum_{j=1}^{r}\frac{\mu^2}{(\psi\mu+j-1)^2}+\sum_{j=1}^{m-r}\frac{(1-\mu)^2}{(\psi(1-\mu)+j-1)^2}\right.\right.$$
$$\left.\left.-\sum_{j=1}^{m}\frac{1}{(\psi+j-1)^2}\right\}\right].$$

Note that ψ is replaced by $1/\phi-1$ in the above expressions.

The extended quasi-likelihood function in this mixture is

$$q(\mu,\phi)=w\left\{y\log\left(\frac{\mu}{1-\mu}\right)+\log(1-\mu)-y\log\left(\frac{y}{1-y}\right)-\log(1-y)\right\}+\frac{1}{2}\log w$$

and the QL's $\hat{\mu}_{QL}$ and $\hat{\phi}_{QL}$ calculated as

$$\hat{\mu}_{QL} = \bar{y},$$

$$\hat{\phi}_{QL} = \frac{1}{m-1}\left\{m\frac{1}{n}\sum d_i - 1\right\}.$$

And $\hat{\phi}_{AQL}$ can be defined as an estimators satisfying (2.2). In this case, the Bartlett adjustment factor is

$$b = \frac{1}{6m}\frac{1-\mu(1-\mu)}{\mu(1-\mu)}\frac{1+(m-1)\phi}{m}.$$

Also, the quasi-expected Fisher information matrix is

$$n\,Var(\hat{\mu}_{QL}) = \frac{\mu(1-\mu)}{w},$$

$$n\,Var(\hat{\phi}_{QL}) = \left(\frac{m}{m-1}\right)^2\frac{2}{w^2}(1+b)^2.$$

The MM's of μ and ϕ are given by equating

$$\mu = \bar{y},$$

$$\frac{1}{m}\mu(1-\mu)(1+(m-1)\phi) = s^2$$

and, hence,

$$\hat{\mu}_{MM} = \bar{y},$$

$$\hat{\phi}_{MM} = \frac{1}{m-1}\left\{\frac{ms^2}{\bar{y}(1-\bar{y})} - 1\right\}.$$

The asymptotic variances of $\hat{\mu}_{MM}$ and $\hat{\phi}_{MM}$ can be obtained by the same method done in Section 2.1. Here, we have,

$$\mu_2 = \mu(1-\mu)/w,$$

$$\mu_3 = \mu(1-\mu)(1-2\mu)/w^2,$$

$$\mu_4 = \mu(1-\mu)(1-6\mu+6\mu^2)/w^3 + 3\mu^2(1-\mu)^2/w^2$$

and therefore,

$$n\,Var(\hat{\mu}_{MM}) \simeq \mu(1-\mu)/w,$$

$$n\,Var(\hat{\phi}_{MM}) \simeq \left(\frac{m}{m-1}\right)^2 \frac{2}{w^2}\left(1-\frac{1}{w}\right)$$

and

$$\text{ARE}(\hat{\mu}_{QL}, \hat{\mu}_{MM}) = 1,$$

$$\text{ARE}(\hat{\phi}_{QL}, \hat{\phi}_{MM}) = \frac{m-1}{m}(1-\phi)\left\{1 + \frac{1}{6m}\frac{1+(m-1)\phi}{m}\frac{1-\mu(1-\mu)}{\mu(1-\mu)}\right\}^{-2}$$

3. SIMULATION

The ARE's of QL with respect to MM are given by simple formula. In fact, they are equal to 1 except ARE($\hat{\phi}_{QL}, \hat{\phi}_{MM}$). However, the ARE's ML with respect to QL or MM should be calculated based on the Fisher information matrix for each μ and ϕ. Also, we must note that the off-diagonal term $E[-\partial^2 l/\partial\mu\partial\phi]$ should be close to zero so that we can use the inverse of the diagonal term as the asymptotic variance. Fortunately, the off-diagonal term in the Fisher information matrix was close to zero for various μ and ϕ.

3.1 Gamma-Poisson mixture

For fixed μ and ϕ, probabilities of negative binomial distribution are calculated. We generate a set of random numbers from that distribution using IMSL(GGDA). Given the data set, ML, QL, and MM of μ and ϕ are calculated, and ARE's of ML with respect to QL(MM) are obtained. We replicate this process 100 times and average them. Simulations are done for $\mu = .2, .5, 1, 2$, and 5 and $\phi = .1(.1).5$. Table 3 list bias of $\hat{\phi}$ and Table 4 list ARE's (each row corresponds to ML, QL, and MM of μ and ϕ, respectively).

For the bias $E(\hat{\theta} - \theta)$, three estimators (in fact, $\hat{\mu}_{QL} = \hat{\mu}_{MM}$) of μ are quite similar and correct. In fact, they are unbiased, and we omit them from the table. $\hat{\phi}_{ML}$ and $\hat{\phi}_{MM}$ are almost equal and good, however, $\hat{\phi}_{QL}$ underestimate when $\mu < 1$ and overestimate when $\mu > 1$. Also, $\hat{\phi}_{AQL}$ always underestimate.

For the ARE of μ, ML is best in all cases. For fixed μ, ARE($\hat{\mu}_{MM}, \hat{\mu}_{ML}$) decreases as ϕ increases, and for fixed ϕ, ARE($\hat{\mu}_{MM}, \hat{\mu}_{ML}$) increases as μ increases. For the ARE of ϕ, ML is best in most cases and QL is worst in all cases.

Conclusively, MM which is easily calculated has small bias and high efficiency for moderate range of μ and ϕ.

Table 3. Bias of $\hat{\phi}_{ML}$, $\hat{\phi}_{MM}$, $\hat{\phi}_{QL}$, and $\hat{\phi}_{AQL}$ in negative binomial distribution

ϕ \ μ	.2	.5	1.	2.	5.
.1	.006	-.001	-.006	-.001	-.003
	.010	-.001	-.005	-.001	-.002
	-.379	-.045	.118	.137	.045
	-.594	-.274	-.063	.030	.004
.2	-.003	-.003	-.002	.001	-.001
	-.003	-.003	-.002	.003	.000
	-.463	-.099	.094	.132	.048
	-.686	-.344	-.106	.010	-.000
.3	.006	.010	-.003	.002	-.001
	.005	.010	-.001	.004	.000
	-.539	-.147	.063	.125	.048
	-.773	-.411	-.156	-.014	-.007
.4	.028	-.080	.005	-.005	-.005
	.021	-.010	.008	-.004	-.004
	-.611	-.218	.035	.109	.044
	-.856	-.494	-.204	-.045	-.020
.5	.001	-.014	.003	-.003	-.003
	.000	-.012	.002	-.003	-.001
	-.706	-.281	-.001	.098	.044
	-.956	-.571	-.258	-.072	-.028

Table 4(a). ARE($\hat{\mu}_{QL}, \hat{\mu}_{ML}$) in negative binomial distribution

ϕ \ μ	.2	.5	1.	2.	5.
.1	.9847	.9924	.9959	.9979	.9991
.2	.9569	.9756	.9859	.9924	.9968
.3	.9272	.9548	.9725	.9847	.9934
.4	.8984	.9326	.9572	.9754	.9893
.5	.8715	.9102	.9409	.9652	.9845

Table 4(b). ARE($\hat{\phi}_{QL}, \hat{\phi}_{ML}$) and ARE($\hat{\phi}_{MM}, \hat{\phi}_{ML}$) in negative binomial distribution

ϕ \ μ	.2	.5	1.	2.	5.
.1	.3936	.6292	.7613	.8549	.9187
	.9643	.9795	.9691	.9702	.9682
.2	.4614	.6746	.8017	.8854	.9431
	.9257	.9456	.9625	.9740	.9808
.3	.5071	.7098	.8240	.8990	.9518
	.8881	.9149	.9398	.9608	.9775
.4	.5378	.7339	.8383	.9064	.9556
	.8553	.8843	.9143	.9432	.9696
.5	.5572	.7504	.8476	.9103	.9570
	.8265	.8561	.8890	.9242	.9597

3.2 Beta-binomial mixture

Same structure of simulation as in Section 3.1 is done for μ = .1(.1).5 and ϕ = .01(.01).05, and m is fixed as 10. Biases and ARE's of three estimators are given in Table 5 and 6, respectively.

For the bias of ϕ, $\hat{\phi}_{QL}$ overestimate in all cases and $\hat{\phi}_{AQL}$ still overestimate even though it mitigate slightly. For the ARE of μ, ML is always best. The ARE($\hat{\mu}_{ML}, \hat{\mu}_{MM}$) is small for small μ and large ϕ. The ARE of ϕ shows different pattern. The ARE($\hat{\phi}_{ML}, \hat{\phi}_{MM}$) decreases as μ and ϕ increase.

Again, as in the negative binomial case, MM performs quite well in the sense of bias and ARE, so we recommend to use MM as estiamtors for μ and ϕ.

Table 5. Bias of $\hat{\phi}_{ML}$, $\hat{\phi}_{MM}$, $\hat{\phi}_{QL}$, and $\hat{\phi}_{AQL}$ in beta-binomial distribution

ϕ \ μ	.1	.2	.3	.4	.5
.01	-.004	-.009	-.004	-.004	-.003
	-.001	-.000	-.000	-.000	-.000
	.017	.017	.011	.008	.008
	.014	.015	.010	.007	.007
.02	-.003	-.002	-.001	.000	-.001
	.000	.000	-.000	.001	-.000
	.016	.018	.013	.011	.009
	.013	.017	.012	.010	.008
.03	.001	-.000	.001	.000	-.000
	.001	.000	.001	.000	.000
	.015	.019	.015	.012	.011
	.011	.017	.014	.011	.010
.04	.000	-.001	-.000	.000	.000
	.000	-.001	-.000	.000	.000
	.012	.019	.016	.014	.013
	.009	.017	.015	.013	.012
.05	.000	.000	-.000	-.000	-.001
	.000	.001	-.000	.000	-.001
	.010	.021	.018	.015	.014
	.006	.018	.016	.014	.013

Table 6(a). ARE($\hat{\mu}_{QL}, \hat{\mu}_{ML}$) in beta-binomial distribution

ϕ \ μ	.1	.2	.3	.4	.5
.01	.9973	.9991	.9997	.9999	1.0000
.02	.9907	.9968	.9989	.9997	1.0000
.03	.9816	.9935	.9977	.9995	1.0000
.04	.9711	.9894	.9962	.9991	.9999
.05	.9598	.9848	.9944	.9987	.9999

Table 6(b). ARE($\hat{\phi}_{QL}, \hat{\phi}_{ML}$) and ARE($\hat{\phi}_{MM}, \hat{\phi}_{ML}$) in beta-binomial distribution

ϕ \ μ	.1	.2	.3	.4	.5
.01	.9042 .9944	.8821 .9978	.8755 .9993	.8726 .9996	.8719 .9999
.02	.9258 .9825	.8757 .9935	.8603 .9977	.8464 .9995	.8524 .9999
.03	.9395 .9687	.8661 .9877	.8438 .9955	.8438 .9989	.8324 .9999
.04	.9477 .9552	.8546 .9814	.8265 .9929	.8154 .9982	.8123 .9998
.05	.9519 .9426	.8418 .9750	.8088 .9903	.7958 .9974	.7922 .9996

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