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STRUCTURE OF THE SPIRAL GALAXY NGC 300

– 1. The generalzation of Toomre's mass model –

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ABSTRACT

In 1963, Toomre built up classes of mass models for the highly flattened galaxies which have free parameters n, a_n and C_n. In order to keep the universal dimension, we adopt parameters $b_n(C_n^2 = a_n^{2n+2}b_n^2/(n-1)!)$ instead of C_n. Series of the normalized Toomre's mass models ($G = V_{max} = R_{max} = 1$, $n = 1$ to 7) are derived and the normalized parameters a_n and b_n are determined by the iteration method. Replacing parameters a_n and b_n to $a'_n(= a_n r_{max})$ and $b'_n(= b_n \cdot V_{max}/r_{max})$, we can get the generalization of Toomre's mass model.

1. INTRODUCTION

The pioneering work of the mass calculation of M31 by Bobcock in 1939 was the first effort to get the mass distribution in spiral galaxies. This work was extended by Wyse, Mayall, Lohmann, Schmidt and Burbidge to get mass models. Brandt and his collegues made mass models after modifying Burbidge *et al.*'s (1959) mass model (Brandt 1960, Brandt and Belton 1962, Brandt and Scheer 1965). Takase and Kinoshita (1967) extended the Brandt's mass model to apply to the rotation energy and the angular momentum.

Toomre (1963) made a new mass model using the Bessel function. His model was consisted with the rotation velocity and the surface density (n, a_n, C_n). The n is defined as a number parameter, and where n is 1 we call it the Toomre's mass model 1 and where n is infinitive then it will be called as a Gaussian model. After Shu (1969) used the Toomre's mass model 1 to the dynamical structure of the Galaxy, many authors have used this model to study the dynamical structure of spiral galaxies. Athanassoula and Sellwood (1986) used the Toomre's mass model 1 to test the instability of the disk, and Johns and Nelson (1986) used this model to the density wave to form spiral arms. Nordsieck (1973 a, b) modified the Toomre's mass model 1 and applied his new model to 17 galaxies to get the angular momentum, radial velocities and integrated masses.

In this paper we normalized 3 parameters (n, a_n, C_n) of the Toomre's mass model and tried to get the total mass, integrated mass, surface density, space density, rotation velocity, angular momentum, rotation energy, angular velocity and the epicyclic frequency. On the generalization of the mass model we changed 3 parameters (n, a_n, C_n) to (n, a'_n, b'_n) where a_n is the distance unit, b_n is the angular velocity unit and n is the parameter to define the rotation curve.

2. GENERALIZATION OF THE TOOMRE'S MASS MODEL

2.1 Toomre's Mass Model

Toomre's mass model assumed the balance of the centrifugal acceleration and the gravitational acceleration in the disk.

$$\frac{V^2(r)}{r} = -\left[\frac{\partial\Phi}{\partial r}\right]_{Z=0} \tag{1}$$

where V(r) is the rotation velocity, r is the radial distance and $\Phi = \phi(r,z)$ is the gravitational potential.

Mass $d\mu(r)$ to the unit area in the thin disk can be defined as

$$d\mu(r) = -\frac{1}{2\pi G}\left[\frac{\partial}{\partial Z}d\Phi\right]_{Z=0} = \frac{k}{2\pi G}J_o(kr)dk \tag{2}$$

where J_o is the Bessel function.

If there does not exist any materials outside the disk, then the gravitational potential equation will satisfy the Poisson equation. So the surface mass density will be as

$$\mu(r) = \int_0^\infty J_o(rk)kS(k)dk \tag{3}$$

where

$$S(k) = \int_0^\infty J_o(ku)u\mu(u)du \tag{4}$$

The real potential can be expressed as equation (5).

$$\Phi(r,z) = 2\pi G\int_0^\infty J_1(rk)S(k)exp(-k \mid z \mid)dk \tag{5}$$

$$-\left[\frac{\partial\Phi}{\partial r}\right]_{Z=0} = 2\pi G\int_0^\infty J_1(rk)kS(k)dk \tag{6}$$

and equation (1) can rewrite as

$$\frac{V^2(r)}{r} = \int_0^\infty J_1(rk)k \int_0^\infty V^2(u)J_1(ku)dudk \tag{7}$$

Using this equation we can get the surface mass density $\mu(r)$ as equation (8).

$$\mu(r) = -\frac{1}{2\pi G}\int_0^\infty J_0(rk)k \int_0^\infty V^2(u)J_1(ku)dudk \tag{8}$$

If we give the boundary condition as $V(0) = V(\infty) = 0$, then equation (8) will be transfer to equation (9) as

$$\mu(r) = \frac{1}{2\pi G}\int_0^\infty \frac{dV^2}{du}H(u,r)du \tag{9}$$

where

$$H(u,r) = \begin{bmatrix} r^{-1}F(1/2,1/2;1;u^2/r^2) = (2/\pi r)K(u/r) \ when \ u < r \\ u^{-1}F(1/2,1/2;1;r^2/u^2) = (2/\pi r)K(r/u) \ when \ u > r \end{bmatrix}$$

The rotation velocity profile can be defined as equation (10).

$$V^2(r) = V_0^2(r) = C_0^2(1 + r^2/a^2)^{-1/2} \tag{10}$$

where C_0 and a are constant values. In this case equation(8) can rewrite as equation (11)

$$\int_0^\infty C_0^2\left[1 + \frac{u^2}{a^2}\right]^{-1/2} J_1(ku)du = C_0^2 a I_{1/2}\left[\frac{1}{2}ka\right] K_{1/2}\left[\frac{1}{2}ka\right] \tag{11}$$

where the Bessel function $I_{1/2}$ and $K_{1/2}$ are defined as

$$I_{1/2}(x) = (2/\pi x)^{1/2}\sinh x,$$
$$K_{1/2}(x) = (\pi/2x)^{1/2}e^{-x}$$

If we put this Bessel Function to the equation (8), then

$$\mu_0(r) = \frac{C_0^2}{2\pi G}\int_0^\infty (1 - e^{-ak})J_0(rk)dk \tag{12}$$

where $\int_0^\infty e^{-ak}J_0(rk)dk = (a^2 + r^2)^{-1/2}$
So we can rewrite the equation (12) as

$$\mu_0(r) = \frac{C_0^2}{2\pi G}\left[\frac{1}{r} - (a^2 + r^2)^{-1/2}\right] \tag{13}$$

Equations (10) and (13) are special functions of the Toomre's mass model, and we can get the Toomre's mass model 1 as

$$
\begin{aligned}
V_1^2(r) &= (C_1^2/a)r^2(a^2+r^2)^{-3/2} \\
\mu_1(r) &= \frac{C_1^2}{2\pi G}(a^2+r^2)^{-3/2}
\end{aligned}
\tag{14}
$$

and the Toomre's mass model n can be expressed as equations (15) and (16).

$$
V_n^2(r) = C_n^2\left[-\frac{\partial}{\partial a^2}\right]^{n-1}\left[\left[\frac{r^2}{a}\right](a^2+r^2)^{-3/2}\right] \tag{15}
$$

$$
\mu_n(r) = \frac{C_n^2}{2\pi G}\left[-\frac{\partial}{\partial a^2}\right]^{n-1}\left[(a^2+r^2)^{-3/2}\right] \tag{16}
$$

2.2 Normalization of the Toomre's Mass Model

As in equations (15) and (16), Toomre's mass model was made with 3 parameters (n, a_n, C_n), and n is a parameter of the decline after maximum value in a rotation velocity curve. The a_n is the distance unit which is correlated with the radial distance (r_{max}) to the maximum rotation velocity (V_{max}). The C_n is a parameter which is connected with V_{max} and r_{max}, and it is changeable to the mass model.

For the practical use we change the C_n to b_n. The derived correlation between C_n and b_n is as equation (17).

$$
C_n^2 = \frac{1}{(n-1)!}\, a_n^{2n+2} \cdot b_n^2 \tag{17}
$$

2.2.1 Rotation velocity and normalization factors a_n and b_n.

We made a differential calculus to the equation (15) from n=1 to n=7 and calculated the an (r_{max}) from the condition of $dV_n(r)/dr = 0$. In models 1 and 2 we can get easily $a_n = a_n(r_{max})$ because the equation $dV_n(r)/dr = 0$ has an analytic solution. In models 3 to 7, however, we get approximate solutions using the Newton-Rapson method. Rotation velocity equations and equations of $dV_n(r)/dr = 0$ from models 1 to 7 are listed in an Appendix A.

For the practical use we changed the equation (15) to equation (18) as

$$
V_n^2(r) = r^2 b^2 \sum_{j=0}^{n-1} \frac{(2n-2j-2)!(2j+1)!}{[j!(n-j-1)!2^{n-1}]^2}(1+r^2/a^2)^{-(j+3/2)} \tag{18}
$$

We plotted rotation velocities of models 1 to 7 in Figure 1 and see that rotation curves decline rapidly to the increased parameter n.

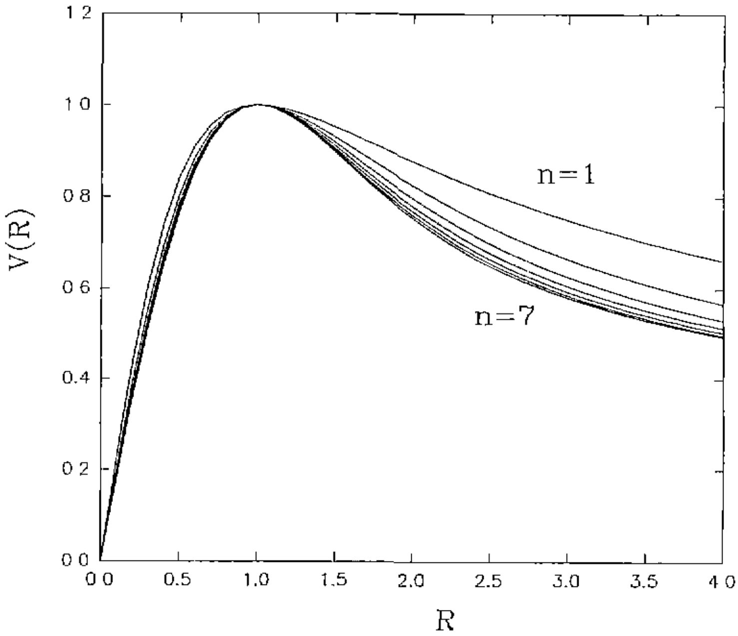

Figure 1. Normalized rotation velocity curves to the radial distance. n is a model parameter to define a shape of curves.

2.2.2 Surface density.

Equation (16) was converted to equation (19) for the practical use as

$$\mu_n(r) = \frac{b^2 a}{2\pi G}\frac{(2n-1)!}{[(n-1)!\,2^{n-1}]^2}(1 + r^2/a^2)^{-(n+1/2)} \tag{19}$$

Appendix B is a list of surface density equations and the surface density curve to the radial distance is plotted in Figure 2. Model 1 has the highest central surface density, while model 7 has the least one.

2.2.3 Space density

Space density was estimated from the solution of the Poisson equation of the Kuzmin-Toomre's potential. Space density equation was made through equations of (20) to (23).

$$\Phi_1(r) = -\frac{GM}{(r^2 + a^2)^{1/2}} \tag{20}$$

$$\Phi_n(r) = -\frac{a^{2n}}{(n-1)!}\left[-\frac{\partial}{\partial a^2}\right]^{n-1}\left[\frac{\Phi_1(r)}{a^2}\right] \tag{21}$$

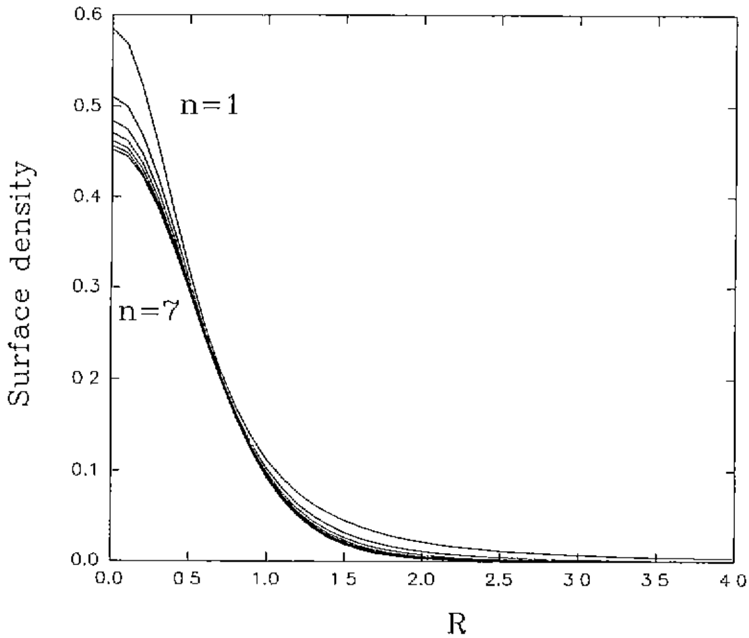

Figure 2. Normalized surface density curves.

$$\nabla^2\Phi_n(r) = -4\pi G\rho'_n(r) \tag{22}$$

where

$$\rho'_n(r) = \frac{a^{2n}\cdot M\cdot(2n+1)!}{\pi(n-1)!\;2^{n-1}}\,(r^2/a^2)^{-(n+3/2)}$$
$$\rho_n(r) = \frac{\rho'_n}{q_o} \tag{23}$$

where $q_o = c/a = (1-e^2)^{1/2}$. Equation (23) is quite similar to a space density equation of Belton and Brandt's (1963). Shu *et al.* (1971) adapted a variable $a = [w^2 + z^2/(1-e^2)]^{1/2}$ to get the space density.

In an Appendix C we listed space density equations and Figure 3 is the space density curves of models 1 to 7.

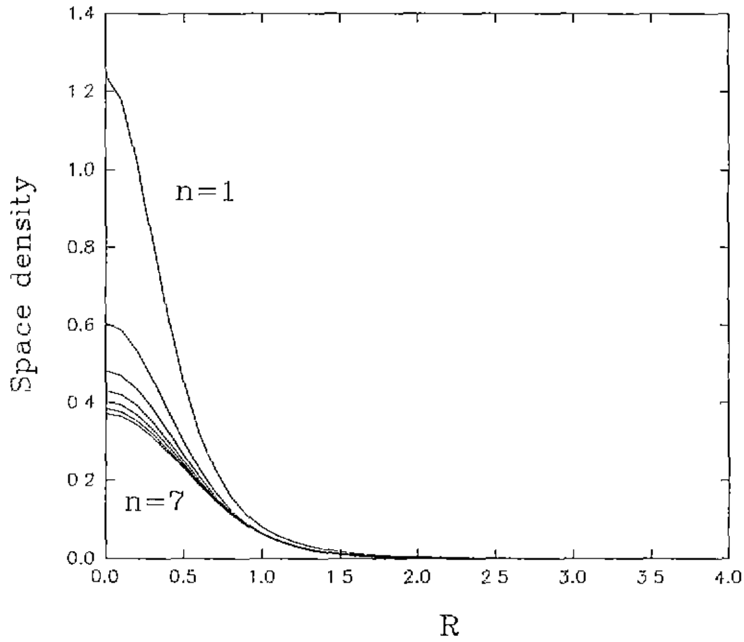

Figure 3. Normalized space density curves.

2.2.4 Total mass

A total mass could be calculated through equations (24) to (26).

$$M_T = \lim_{r\to\infty} \frac{r\ V_n^2(r)}{G} \tag{24}$$

where

$$\lim_{r\to\infty} V_n^2(r) = \lim_{r\to\infty} C_n^2 \left[-\frac{\partial}{\partial a^2}\right]^{n-1} \frac{1}{ar(a^2/r^2+1)^{3/2}}$$
$$= C_n^2 \left[-\frac{\partial}{\partial a^2}\right]^{n-1} \frac{1}{ar}$$

$$M_T = \frac{C_n^2}{G} \left[-\frac{\partial}{\partial a^2}\right]^{n-1} \frac{1}{a}$$
$$= \frac{C_n^2}{G} \prod_{m=2}^{n} \frac{1}{2}(2m-3)\cdot(a^2)^{-(n-1/2)} \tag{25}$$

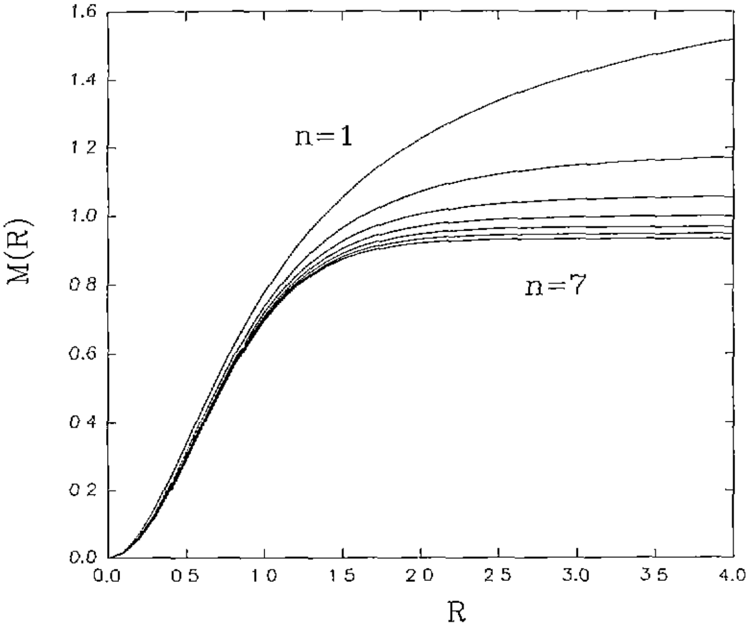

Figure 4. Normalized integrated mass curves.

where

$$\prod_{m=2}^{n} \frac{1}{2}(2m-3) = \frac{(2n-2)!}{(n-1)!\ 2^{2n-2}}$$

If we replace C_n to b_n, then we can get the equation (26) as

$$M_T = \frac{(2n-2)!}{[(n-1)!\ 2^{2n-1}]^2 G}\ b^2 \cdot a^3 \tag{26}$$

Model 1 has about 2 time more total mass than model 7 ($M_7/M_1 = 0.5071$). Appendix D is total mass equations.

2.2.5 Integrated mass

The integrated mass within a radial distance r from the galactic centre can be described as

$$M(r) = \int_0^{\infty} 2\pi r \mu(r) dr$$
$$= \frac{b^2[a^3(a^2+r^2)^{(2n-1)/2} - a^{2(n+1)}](2n-1)!}{2^{2(n-1)}G(2n-1)(a^2+r^2)^{(n-1/2)}[(n-1)!]^2} \quad (27)$$

We listed integrated mass equations in Appendix E and plotted integrated mass curves to the radial distance in Figure 4.

Ratio of the integrated mass to the total mass $M(r)/M_T$ is a good parameter to indicate the central mass concentration. In Figure 5 we plotted this ratio to the radial distance. From this figure we can see that the more n value, the higher central concentration.

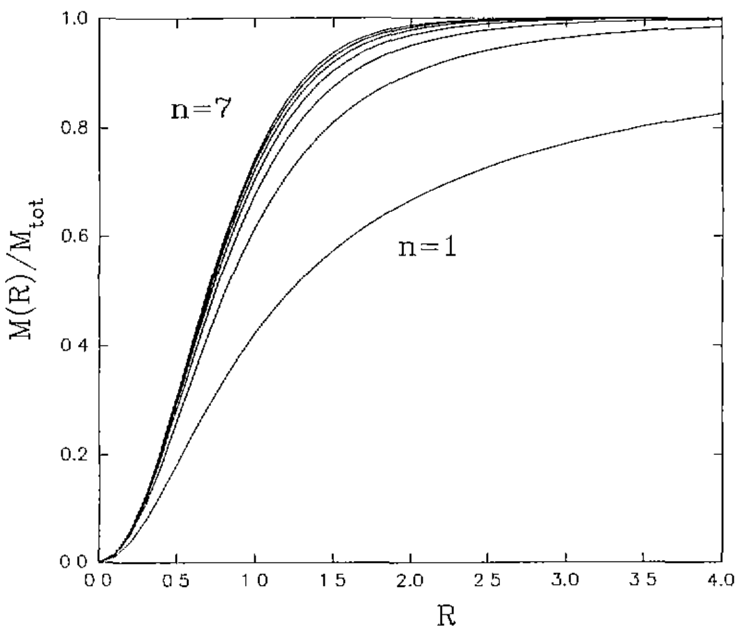

Figure 5. Ratios of the integrated mass to the total mass curves.

2.2.6 Angular momentum

Angular momentum at the radial distance r from a galactic centre is defined as

$$H(r) = \int_0^r rV(r)\, dM(r) = 2\pi \int_0^r r^2 V(r)\mu(r)dr \tag{28}$$

To get the total angular momentum we used an equation suggested by Saslaw (1985) as

$$H_T = \left[\frac{G^2}{\pi\mu_n(0)}\right]^{1/4} M_T^{7/4} \tag{29}$$

where $\mu_n(0)$ is a central surface density and M_T is a total mass. We listed total angular momentum equations in Appendix F and plotted this to the model parameter n in Figure 6.

2.2.7 Rotation energy

Rotation energy of a galaxy can be calculated through an equation (30) as

$$E(r) = \frac{1}{2}\int_0^r V^2(r)dM(r) = \pi \int_0^r rV^2(r)\mu(r)dr \tag{30}$$

Total rotation energy was estimated from the integration of equation (31) as

$$E_T = \pi \int_0^r rV^2(r)\mu(r)dr \tag{31}$$

We listed total rotation energy equations to the model parameter n in an Appendix G and plotted this in Figure 7.

2.2.8 Centrifugal force

The centrifugal force F(r) can be described as equation (32) as

$$F(r) = \frac{V^2(r)}{r} = rb^2 \sum_{j=0}^{n-1} \frac{(2n-2j-2)!(2j+1)!}{[j!(n-j-1)!2^{n-1}]^2}(1+r^2/a^2)^{-(j+3/2)} \tag{32}$$

We plotted the generalized centrifugal force curve in Figure 8.

2.2.9 Angular velocity and epicyclic frequency

An epicyclic frequency $\kappa(r)$ is expressed in equation (33) as

$$\kappa^2(r) = 4\Omega^2(r)\left[1 + \frac{r}{2\Omega(r)}\frac{d\Omega(r)}{dr}\right] \tag{33}$$

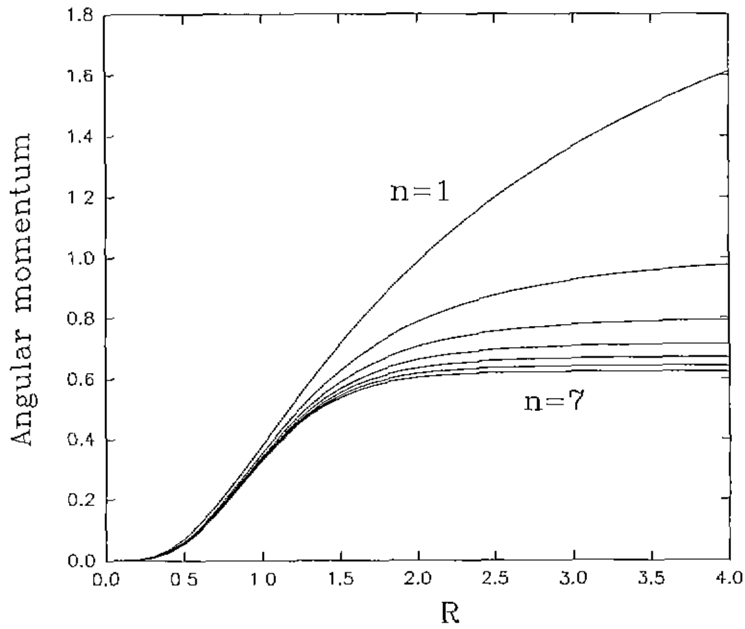

Figure 6. Normalized angular momentum curves.

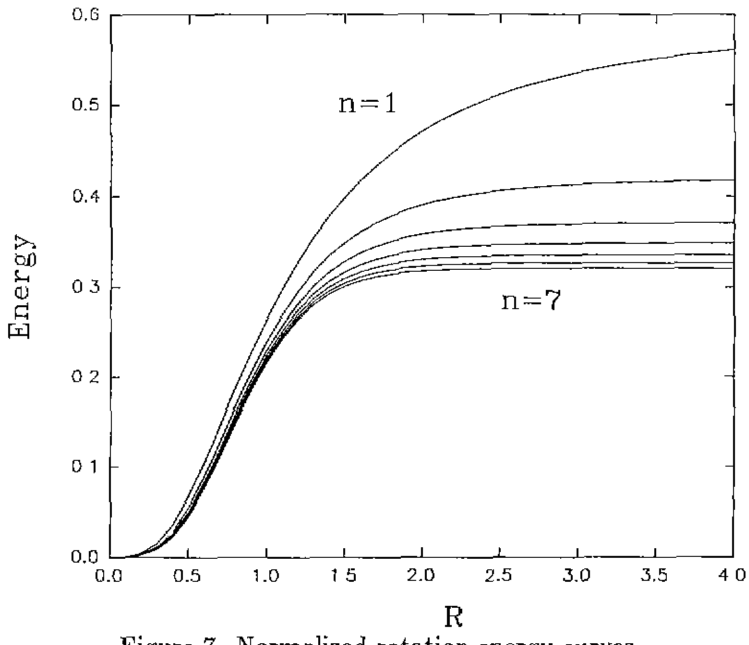

Figure 7. Normalized rotation energy curves.

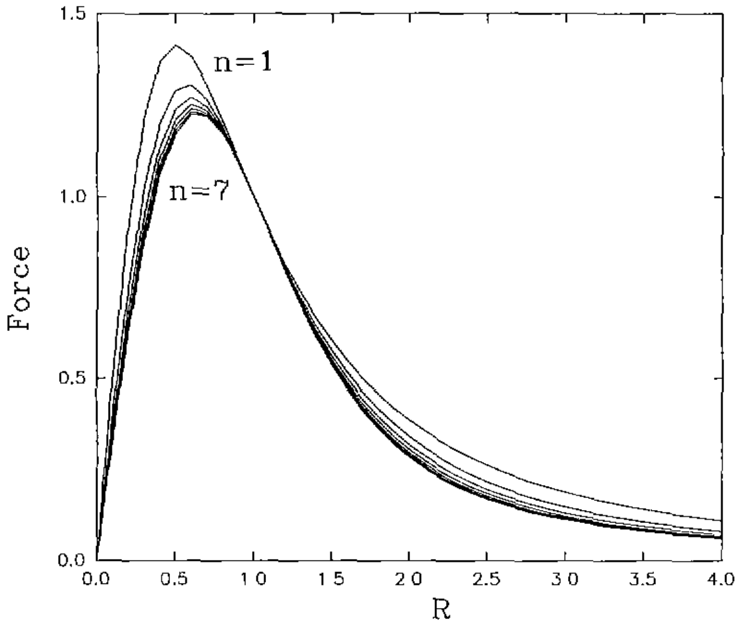

Figure 8. Normalized centrifugal force curves.

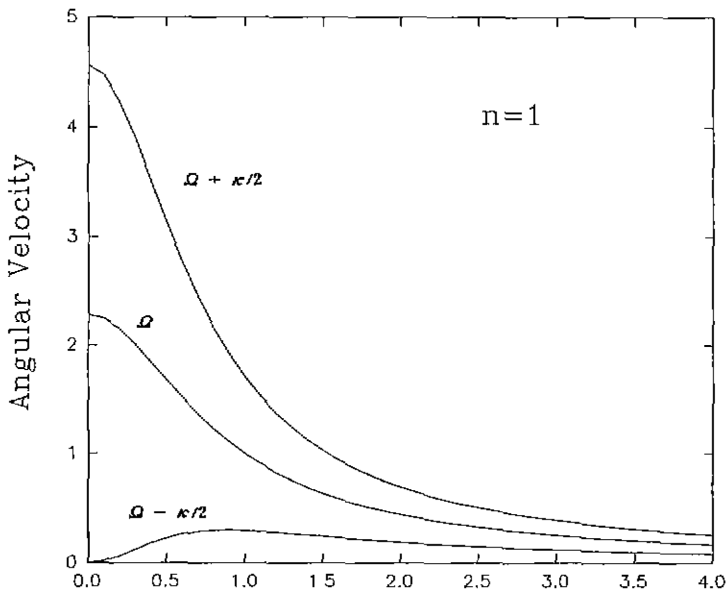

Figure 9. Angular velocity and the Lindblad resonance curves of the normalized mass model 1.

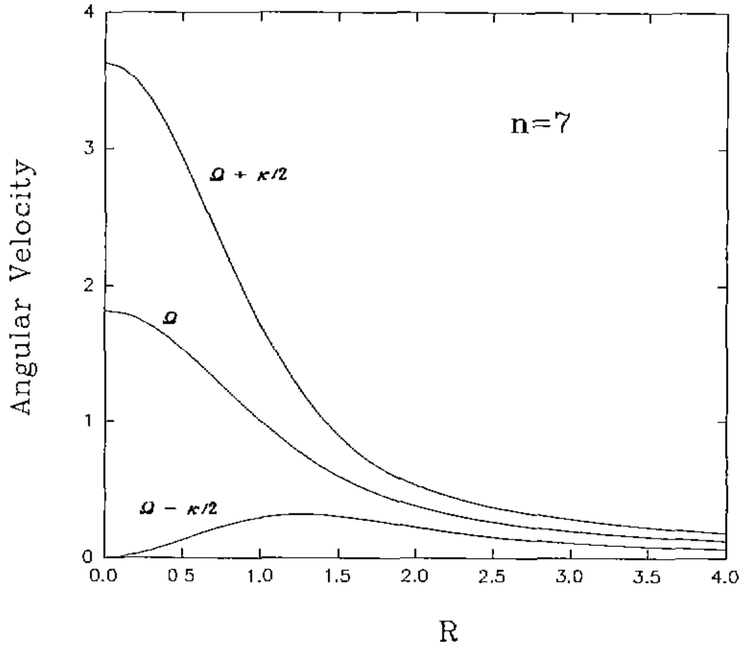

Figure 10. Same as in Figure 9 of the case of model 7.

where $\Omega(r)$ is an angular velocity. We listed epicyclic frequency equations in an Appendix H.

Figure 9 and Figure 10 are plottings of angular velocity $\Omega(r)$ curves and the Lindblad resonance $\Omega(r) \pm \kappa/2$ curves to model parameters 1 and 7, respectively.

3. CONCLUSION

All equations after the normalization of the Toomre's mass model can be defined from 3 parameters (n, a_n, b_n). If we change normalization values V_{max}, r_{max} and G to observed values, then a_n and b_n can be rewrited as

$$a'_n = a_n \cdot r_{max}$$

$$b'_n = b_n \cdot \frac{V_{max}}{r_{max}}$$

The rotation velocity curve, which can be obtained from the optical and radio observations, will be fitted to get the model parameter n. A polynomial fitting to the rotation curve will be possible to estimate V_{max}, r_{max} and 3 paremetrs (n, a_n, b_n). Using these parameters we can calculate physical and dynamical values of a sample galaxy.

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Appendix A; Rotation velocity equations to the model parameter n.

$$V_1^2(r) = r^2 b_1^2 (1 + r^2/a_1^2)^{-3/2}$$
$$a_1 = r_{max}/\sqrt{2} \qquad (A1)$$

$$V_2^2(r) = \frac{r^2 b_2^2 a_2^3 (r^2 + 4a_2^2)}{2(r^2 + a_2^2)^{5/2}}$$
$$a_2 = \frac{(2^{3/2} + 2 \cdot 3^{1/2})^{1/2}}{2^{5/4}} \cdot r_{max} \qquad (A2)$$

$$V_3^2(r) = \frac{3r^2 b_3^2 a_3^3 (4r^2 a_3^2 + r^4 + 8a_3^4)}{8(r^2 + a_3^2)^{7/2}}$$
$$a_3 = -24r_{max}^2 a_3^4 - 6r_{max}^4 a_3^2 - r_{max}^6 + 16a_3^6 = 0 \qquad (A3)$$

$$V_4^2(r) = \frac{r^2 b_4^2 a_4^3 (48r^2 a_4^4 + 24r^4 a_4^2 + 5r^6 + 64a_4^6)}{16(r^2 + a_4^2)^{9/2}}$$
$$a_4 = -256r_{max}^2 a_4^6 - 96r_{max}^4 a_4^4 - 32r_{max}^6 a_4^2 - 5r_{max}^8 + 128a_4^8 = 0 \qquad (A4)$$

$$V_5^2(r) = \frac{5r^2 b_5^2 a_5^3 (128r^2 a_5^6 + 96a_5^4 r^4 + 40r^6 a_5^2 + 7r^8 + 128a_4^8)}{128(r^2 + a_5^2)^{11/2}}$$
$$a5 = -640r_{max}^2 a_5^8 - 320r_{max}^4 a_5^6 - 160r_{max}^6 a_5^4$$
$$- 50r_{max} 8a_5^2 - 7r_{max}^{10} + 256a_5^{10} = 0 \qquad (A5)$$

$$V_6^2(r) = \frac{3r^2 b_6^2 a_6^3 (640r^2 a_6^8 + 640r^4 a_6^6 + 400r^6 a_6^4 + 140r^8 a_6^2 + 21r^{10} + 512a_6^{10})}{256(r^2 + a_6^2)^{13/2}}$$
$$a6 = -3072r_{ax}^2 a_6^2 - 1920r_{max}^4 a_6^8 - 1280r_{max}^6 a_6^6 - 600r_{max}^8 a_6^4$$
$$- 168r_{max}^{10} a_6^2 - 21r_{max}^{12} + 1024a_6^{12} = 0 \qquad (A6)$$

$$V_7^2(r) = \frac{\begin{array}{l} 7r^2 b_7^2 a_7^3 (1536r^2 a_7^{10} + 1920r^4 a_7^8 + 1600r^6 a_7^6 \\ \qquad +840r^8 a_7^4 + 252r^{10} a_7^2 + 33r^{12} + 1024 + a_7^{12}) \end{array}}{1024(r^2 + a_7^2)^{15/2}}$$
$$a7 = -7168r_{max}^2 a_7^{12} - 5376r_{max}^4 a_7^{10} - 4480r_{max}^6 a_7^8$$
$$- 2800r_{max}^8 a_7^6 - 1176r_{max} 10a_7^4 - 294r_{max}^{12} a_7^2 - 33r_{max}^{14} + 2048a_7^{14} = 0 \qquad (A7)$$

Appendix B; Surface density equations

$$\mu_1(r) = \frac{a_1 b_1^2}{2\pi G}\,(1 + r^2/a_1^2)^{-3/2} \tag{B1}$$

$$\mu_2(r) = \frac{3a_2 b_2^2}{4\pi G}\,(1 + r^2/a_1^2)^{-5/2} \tag{B2}$$

$$\mu_3(r) = \frac{15a_3 b_3^2}{16\pi G}\,(1 + r^2/a_3^2)^{-7/2} \tag{B3}$$

$$\mu_4(r) = \frac{35a_4 b_4^2}{32\pi G}\,(1 + r^2/a_4^2)^{-9/2} \tag{B4}$$

$$\mu_5(r) = \frac{315a_5 b_5^2}{256\pi G}\,(1 + r^2/a_5^2)^{-11/2} \tag{B5}$$

$$\mu_6(r) = \frac{693a_6 b_6^2}{512\pi G}\,(1 + r^2/a_6^2)^{-13/2} \tag{B6}$$

$$\mu_7(r) = \frac{3003a_7 b_7^2}{2048\pi G}\,(1 + r^2/a_7^2)^{-15/2} \tag{B7}$$

Appendix C; Space density equations

$$\rho_1(r) = \frac{3a_1^2 M_1}{4\pi qo}(r^2 + a_1^2)^{-5/2} \tag{C1}$$

$$\rho_2(r) = \frac{15a_2^4 M_2}{8\pi qo}(r^2 + a_2^2)^{-7/2} \tag{C2}$$

$$\rho_3(r) = \frac{105a_3^6 M_3}{32\pi qo}(r^2 + a_3^2)^{-9/2} \tag{C3}$$

$$\rho_4(r) = \frac{945a_4^8 M_4}{192\pi qo}(r^2 + a_4^2)^{-11/2} \tag{C4}$$

$$\rho_5(r) = \frac{1039a_5^{10} M_5}{1536\pi qo}(r^2 + a_5^2)^{-13/2} \tag{C5}$$

$$\rho_6(r) = \frac{135135a_6^{12} M_6}{15360\pi qo}(r^2 + a_6^2)^{-15/2} \tag{C6}$$

$$\rho_7(r) = \frac{2027025 a_7^{14} M_7}{184320 \pi q o}(r^2 + a_7^2)^{-17/2} \tag{C7}$$

Appendix D; Total mass equations

$$M_T(1) = \frac{a_1^3 b_1^2}{G} \tag{D1}$$

$$M_T(2) = \frac{a_2^3 b_2^2}{2G} \tag{D2}$$

$$M_T(3) = \frac{3a_3^3 b_3^2}{8G} \tag{D3}$$

$$M_T(4) = \frac{5a_4^3 b_4^2}{16G} \tag{D4}$$

$$M_T(5) = \frac{35a_5^3 b_5^2}{128G} \tag{D5}$$

$$M_T(6) = \frac{63a_6^3 b_6^2}{256G} \tag{D6}$$

$$M_T(7) = \frac{231a_7^3 b_7^2}{1024G} \tag{D7}$$

Appendix E; Integrated mass equations

$$M_1(r) = b_1^2 a_1^3(-a_1 + (a_1^2 + r^2)^{1/2})/(G(a_1^2 + r^2)^{1/2}) \tag{E1}$$

$$M_2(r) = b_2^2 a_2^3(-a_2^3 + (a_2^2 + r^2)^{3/2})/(2G(a_2^2 + r^2)^{3/2}) \tag{E2}$$

$$M_3(r) = 3b_3^2 a_3^3(-a_3^5 + (a_3^2 + r^2)^{5/2})/(8G(a_3^2 + r^2)^{5/2}) \tag{E3}$$

$$M_4(r) = 5b_4^2 a_4^3(-a_4^7 + (a_4^2 + r^2)^{7/2})/(16G(a_4^2 + r^2)^{7/2}) \tag{E4}$$

$$M_5(r) = 35b_5^2 a_5^3(-a_5^9 + (a_5^2 + r^2)^{9/2})/(128G(a_5^2 + r^2)^{9/2}) \tag{E5}$$

$$M_6(r) = 63b_6^2a_6^3(-a_6^{11} + (a_6^2 + r^2)^{11/2})/(256G(a_6^2 + r^2)^{11/2}) \tag{E6}$$

$$M_7(r) = 231b_7^2a_7^3(-a_7^{13} + (a_7^2 + r^2)^{13/2})/(1024G(a_7^2 + r^2)^{13/2}) \tag{E7}$$

Appendix F; Total angular momentum equations

$$M_T(1) = \frac{2^{1/4}}{G}a_1^5b_1^3 \tag{F1}$$

$$M_T(2) = \left[\frac{4}{3}\right]^{1/4}\left[\frac{2}{1}\right]^{7/4}\frac{1}{G}a_2^5b_2^3 \tag{F2}$$

$$M_T(3) = \left[\frac{16}{15}\right]^{1/4}\left[\frac{3}{8}\right]^{7/4}\frac{1}{G}a_3^5b_3^3 \tag{F3}$$

$$M_T(4) = \left[\frac{32}{35}\right]^{1/4}\left[\frac{5}{16}\right]^{7/4}\frac{1}{G}a_4^5b_4^3 \tag{F4}$$

$$M_T(5) = \left[\frac{256}{315}\right]^{1/4}\left[\frac{35}{128}\right]^{7/4}\frac{1}{G}a_5^5b_5^3 \tag{F5}$$

$$M_T(6) = \left[\frac{512}{693}\right]^{1/4}\left[\frac{63}{256}\right]^{7/4}\frac{1}{G}a_6^5b_6^3 \tag{F6}$$

$$M_T(7) = \left[\frac{2048}{3003}\right]^{1/4}\left[\frac{231}{1024}\right]^{7/4}\frac{1}{G}a_7^5b_7^3 \tag{F7}$$

Appendix G; Total rotation energy equations

$$E_T(1) = \frac{a_1^5b_1^4}{8G} \tag{G1}$$

$$E_T(2) = \frac{5a_2^5b_2^4}{64G} \tag{G2}$$

$$E_T(3) = \frac{63a_3^5b_3^4}{102G} \tag{G3}$$

$$E_T(4) = \frac{429 a_4^5 b_4^4}{8192G} \tag{G4}$$

$$E_T(5) = \frac{12155 a_5^5 b_5^4}{262144G} \tag{G5}$$

$$E_T(6) = \frac{88179 a_6^5 b_6^4}{2097152G} \tag{G6}$$

$$E_T(7) = \frac{1300075 a_7^5 b_7^4}{33554432G} \tag{G7}$$

Appendix H; Epicycle frequency equations to the model parameter n.

$$\kappa_1^2(r) = \Omega_1^2(r)\left[\frac{4 + r^2/a_1^2}{1 + r^2/a_1^2}\right] \tag{H1}$$

$$\kappa_2^2(r) = \Omega_2^2(r)\left[\frac{r^4 + 2a_2^2 r^2 16a_2^4}{r^4 + 5a_2^2 r^2 + 4a_2^4}\right] \tag{H1}$$

$$\kappa_3^2(r) = \Omega_3^2(r)\frac{(r^6 + 32a_3^6 + 4r^4 a_3^2)}{(4r^2 + a_3^2 r^4 + 8a_3^4)(r^2 + a_3^2)} \tag{H3}$$

$$\kappa_4^2(r) = \Omega_4^2(r)\frac{(256a_4^8 + 5r^8 + 48r^4 a_4^4 - 32r^2 a_4^6 + 26r^6 a_4^2)}{(48r^2 a_4^4 + 24r^4 a_4^2 + 5r^6 + 64a_4^6)(r^2 + a_4^2)} \tag{H4}$$

$$\kappa_5^2(r) = \Omega_5^2(r)\frac{(7r^{10} + 512a_5^{10} + 128r^4 a_5^6 + 112r^6 a_5^4 + 44r^8 a_5^2 - 128a_5^8 r^2)}{(128r^2 a_5^6 + 96a_5^4 r^4 + 40r^6 a_5^2 + 7r^8 + 128a_5^8)(r^2 + a_5^2)} \tag{H5}$$

$$\kappa_6^2(r) = \Omega_6^2(r)\frac{(2048a_6^{12} + 21r^{12} + 640r^4 a_6^8 - 768a_6^{10} r^2 + 800r^6 a_6^6 + 480r^8 a_6^4 + 154r^{10} a_6^2)}{(640r^2 a_6^8 + 640r^4 a_6^6 + 400r^6 a_6^4 + 140r^8 a_6^2 + 21r^{10} + 512a_6^{10})(r^2 + a_6^2)} \tag{H6}$$

$$\kappa_7^2(r) = \Omega_7^2(r)\frac{\begin{matrix}(4096a_7^{14} + 33r^{14} + 1536r^4 a_7^{10} - 2048a_7^{12} r^2 \\ +2560r^6 a_7^8 + 2080r^8 a_7^6 + 1008r^{10} a_7^4 + 276r^{12} a_7^2)\end{matrix}}{\begin{matrix}(1536a_7^{10} + 1920r^4 a_7^8 + 1600r^6 a_7^6 + 84f0r^8 a_7^4 + 252r^{10} a_7^2 \\ + 33r^{12} + 1024a_7^{12})(r^2 + a_7^2)\end{matrix}} \tag{H7}$$