다분포 대형 시뮬레이션 모형에 대한 결합상관방법

Combined Correlation Methods for Multipopulation Metamodel

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Abstract

This research develops two variance reduction methods for estimating the parameters of the experimental simulation model having multiple design points based on an approach focusing on reduction of the variances of the mean responses across multiple design points. The first method extends a combined approach of antithetic variates and control variates for a single design point to the multipopulation context with independent streams across the design points. The second method extends the same strategy in conjunction with the Schruben-Margolin method for improving the first method. We illustrate an example for implementing the second method. We expect these two approaches may improve the estimation of the parameters of interest compared with the control variates method.

1. Introduction

Consider the simulation experiment for investigating the effects of the factor variables on the univariate response of interest. To explore the response surface over a factor region of interest, typically we assume that there exists some linear functional relationship between the response and the experimental variables: factor variable and concomitant variable (control variate). Factor variables are under the control of an experimenter, in that we assume that the experimenter can select and set levels of a factor variable without error. In contrast, the levels of the concomitant variables are not set by the experimenter but merely observed in the course of conducting the experiment. That is, a concomitant variable is observed randomly at each of the levels of the factor variables during the experiment and assumed to be correlated with the corresponding response.

For a designed simulation experiment having multiple design points and univariate response

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(Nozari, Pegden and Arnold[8] referred to it as a multipopulation simulation model), several authors have developed procedures that improve the reliability of the estimators for the model parameters. Schruben and Margolin[11] developed a method for combining the use of common random numbers and antithetic variates in one simulation experiment designed to estimate the parameters for the first-order multipopulation model where a design matrix admits orthogonal blocking into two blocks. Nozari, Arnold and Pedgen[8] added control variates to the linear model of factor variables and evaluated the simulation efficiency of control variates in estimating the parameters of the linear model. Tew and Wilson [14] proposed a combined approach using the Schruben-Margolin correlation induction strategy in conjunction with control variates to improve the estimation of the parameters in the first-order linear model.

These studies exploit the correlations between (a) the responses at different design points, and (b) the response and control variates within a design point. Different from these approaches, this paper focuses on reduction in variance of the mean response of interest at a single design point and additionally tries to utilize the correlation between the responses across the design points. We consider that the responses with reduced variances at the design points of the experimental model may ensure improvement in the estimation of the parameters of the multipopulation model.

For a simulation experiment of a single population model, usually antithetic variates and control variates are applied to reduce the error of the estimator for the mean response of interest. Antithetic variates utilize the negative correlation between the responses from different replications. In contrast to the approach of antithetic variates, the method of control variates attempts to exploit correlations between the response and selected control variates within a single run. (see the discussions of these two methods in Kleijnen.[5]) Suppose that through correlated replications of simulation runs at a single design point, we reduce the variance of the estimator for the mean response and yet maintain the same correlation between the response and control variates as those obtained under independent replications. Then we may take advantage of both antithetic variates and control variates together in one simulation run, and reduce the variance of the estimator for the mean response further than by only applying control variates.

Based on this conjecture, we propose a method which focuses on reducing the variances of the mean responses across the design points for estimating the parameters of interest in the multipopulation context. We also consider a strategy incorporating the correlations between the responses at different design points for further improving the precision of the parameter estimation.

2. Notation and Background

Consider an experimental design that specifies the combination of m factor settings in the multipopulation simulation model. Suppose we estimate the mean response at a single design point i by the sample mean, $\bar{y}_i$, of $2h$ replicates simulation runs. Let $\bar{y}=(\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_m)'$ be the mean response vector across the m design points. Suppose that the relationship between the responses and the function of factor settings across all m design points can be written as the following linear model:

$$\bar{y}=X\beta+\varepsilon, \tag{1}$$

where $\bar{y}=(\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_m)'$ is the $(m\times 1)$ vector of responses, $\beta=(\beta_0, \beta_1, \cdots, \beta_p)'$ is the $((p+1)\times 1)$ vector of unknown model coefficients, X is a $(m\times(p+1))$ design matrix whose first column is the $(m \times 1)$ vector of 1's (1_m) and whose ith column consists of levels of the factor variables across m design points, and ε is a $(m \times 1)$ vector which represents the inability of the postulated model to determine $\bar{y}$.

During the simulation experiment, often we observe control variates that are highly correlated with the response of interest. Let $\bar{c}_i$ be the vector of control variates corresponding to $\bar{y}_i$ at design point i. When the length of simulation run is sufficiently large, it is assumed that

$$(y_i, c'_i)' \sim \text{IID } N_{s+1}((\mu_{y_i}, 0')', \Sigma), \quad i = 1, 2, \ldots, m, \tag{2}$$

where y_i and c_i are the response and the $(s \times 1)$ vector of control variates, respectively, at the ith design point, μ_{y_i} is the mean response of the ith design point,

$$\Sigma = \begin{bmatrix} \sigma_y^2 & \sigma'_{yc} \\ \sigma_{yc} & \Sigma_c \end{bmatrix}, \tag{3}$$

is the covariance matrix of the response and control variates, σ_{yc} is the covariance between y_i and c_i, and Σ_c is the covariance matrix of c_i (see the discussions in Lavenberg, Moeller and Welch,[6] Cheng and Feast [3] and Nozari, Pegden and Arnold[8]. Under this assumption, by adding the control variates to (1), the responses of interest at the m design points can be represented as the following linear model:

$$\bar{y} = X\beta_G + \bar{C}\alpha + \varepsilon^{*}, \tag{4}$$

where $\bar{y}$, X and β_G are given in (1), $\bar{C}$ is a $(m \times s)$ matrix whose ith row consists of $\bar{c}'_i$, $\varepsilon = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m)'$ is the $(m \times 1)$ vector of error terms. The least squares estimators of β_G and α in the linear model in (4) are given by, respectively,

$$\hat{\alpha} = (\bar{C}'P\bar{C})^{-1} \text{ and } \hat{\beta}_G = (X'X)^{-1}X'(\bar{y} - \bar{C}\hat{\alpha})$$
$$\text{with } P = I_m - X(X'X)^{-1}X'^{\,(2)}. \tag{5}$$

Nozari, Arnold and Pegden[8] considered the effect of the control variates method by comparing the variance of the least squares estimator β in (1) with that of β_G in (5) under the assumption in (2), and showed that the effect of control variates is

$$Var(\hat{\beta}) - Var(\hat{\beta}_G) = \sigma_y^2\left(1 - \frac{m-p-2}{m-p-s-2}R^2\right)(X'X)^{-1} \quad \text{if } m > p + s + 2, \tag{6}$$

where $R^2 = \sigma'_{yc}\Sigma_c^{-1}\sigma_{yc}$ is the square of the multiple correlation coefficient between y_i and c_i. This result implies that the loss factor of control variates method is $(m-p-2)/(m-p-s-2)$ due to the estimation of α and the minimum variance ratio is R^2.

3. Combined Methods

This section presents two variance reduction methods for estimating the parameters of interest in the multipopulation model. These two methods are based on combined method utilizing antithetic variates and control variates simultaneously to reduce the variance of the mean response at a single design point.

One of the characteristics of the computer simultion is the experimenter's control over the random number streams that drive a simulation model. These streams completely determine the simulation response output. Let R_{ij} be the set of g random number streams for the jth replication of simulation run at the ith design point:

$$R_{ij} = (r_{ij1}, r_{ij2}, \ldots, r_{ijg}) \text{ for } i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, 2h,$$

where $r_{ijk}(k=1, 2, \cdots, g)$ denotes the sequence of random numbers used for driving the kth stochastic component of the simulation model at the ith design point and jth replication.

We first consider the random number assignment strategy of jointly utilizing antithetic variates and control variates for a simulation model which requires g such random number streams to drive all of its stochastic components across replications at a single design point. To this end, we separate R_{ij} into towo mutually exclusive and exhaustive subsets of random number streams, (R_{ij1}, R_{ij2}). We use the first subset of $(g-s^*)$ streams, R_{ij1}, for driving the non-control variate stochastic components, the second subset of s^* streams, R_{ij2}, for driving the control variate stochastic components. The correlated replication strategy at a single design point uses antithetic variates for all stochastic components except the control variates through $2h$ replications. That is, within the jth paired replications, we use $(R_{i2j-1,1}, R_{i2j-1,2})$ and $(\bar{R}_{i2j-1,1}, R_{i2j,2})$, where $R_{i2j-1,1}$, $R_{i2j-1,2}$, and $R_{i2j,-1,2}$ are sets of randomly selected random number streams, and $\bar{R}_{i2j-1,1}$ is antithetic to $R_{i2j-1,1}$. Across pairs of replications, we use independent streams. The complelte assignment of random number streams across $2h$ replications is given in Table 1.

Table 1. Random Number Assignment Rule for a single Design point

Replication	Control Variates	Response
1	$c_1(R_{12})$	$y_1(R_{11}, R_{12})$
2	$c_2(R_{22})$	$y_2(\bar{R}_{11}, R_{22})$
3	$c_3(R_{32})$	$y_3(R_{21}, R_{32})$
4	$c_4(R_{42})$	$y_4(\bar{R}_{21}, R_{42})$
.	.	.
.	.	.
.	.	.
2h-1	$C_{2h-1}(R_{2h-1,2})$	$y_{2h-1}(R_{h-1,1}, R_{2h-1,2})$
2h	$C_{2h}(R_{2h-1,2})$	$y_{2h}(R_{h-1}, R_{2h,2})$

Then, the jth pair of responses at the ith design point, y_{ij2-1} and y_{i2j} $(j=1, 2, \cdots, h)$, are negatively correlated by antithetic streams through the non-control stochastic components. However, through the $2h$ replications, the control variates c_i $(i=1, 2, \cdots, 2h)$ are independently generated by the assignment of independent streams through the control variate stochastic components at each replication. We hypothesize that the response y_{i2j-1} (y_{i2j}) is independent of control variates $c_{i2j}(c_{i2j-1})$ within a paired simulation output due to independent streams for the control variates.

Next we consider two random number assignment strategies across m design points. The first method extends the above mentioned approach for a single population experiment to the multipopulation context with independent random number streams (Combined Method I). The second method uses the same policy as the first method for the experiment at each design point, and additionally uses Schruben Margolin's assignment rule [11] for the experiment across the design points (Combined Method II).

3.1. Combined Method I

Let y_{ij} and c_{ij} be the response of interest and a vector of control variates, respectively, at the jth replication and the ith design point under the replication rule of this method. Based on the earlier duscussions and statistical modeling assumptions, we establish the assumptions for the response and control variates obtained across the m design points and $2h$ replications as follows:

1. $\mathrm{Var}(y_{ij})=\sigma^2_y$, for $i=1, 2, \cdots, m$, $j=1, 2, \cdots, 2h$ (homogeneity of response variances across

design points and replicates),

2. $\mathrm{Cov}(y_{ij}, y_{ik}) = -\rho_y\sigma^2_y (\rho_y > 0)$ for $i=1, 2, \cdots, m$ if $k=j+1 (j=1, 3, \cdots, 2h-1)$ (homogeneity of induced negative correlations across design points and replicates pairs). Otherwise, $\mathrm{Cov}(y_{ij}, y_{ik})=0$,
3. $\mathrm{Cov}(y_{ij}, c_{kl}) = \sigma'_{yc}$ if $i=k$ and $j=l$. (homogeneity of control variates-response covariance across design points and replicates). Otherwise, $\mathrm{Cov}(y_{ij}, c_{kl})=0'$,
4. $\mathrm{Cov}(c_{ij}) = \Sigma_c$, for $i=1, 2, \cdots, m$ and $j=1, 2, \cdots, 2h$. (homogeneity of control variates covariance structure across design points and replicates), and
5. $\mathrm{Cov}(c_{ij}, c_{kl}) = O_{sxs}$ for $i \neq k$ and $j \neq l$(independence of control variates across design points and replicates).

Under these assumptions, we identify the joint distribution of $\bar{y}$ and $\bar{C}$: $\bar{y}=(\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_m)'$ and $\bar{C}=(\bar{c}_1, \bar{c}_2, \cdots, \bar{c}_m)'$. At the ith design point, from Corollary 5.2.1 in Mood, Graybill and Boes [4], the variance of $\bar{y}_i$, is given by

$$\begin{aligned}\mathrm{Var}(\bar{y}_i) &= \mathrm{Var}(\sum_{j=1}^{2h} y_{ij}/2h) = [\sum_{j=1}^{2h}\mathrm{Var}(y_{ij}) \\ &\quad + 2\sum_{j<k}\mathrm{Cov}(y_{ij}, y_{ik})]/4h^2 \\ &= [\sum_{j=1}^{2h}\mathrm{Var}(y_{ij}) + 2\sum_{j=1}^{h}\mathrm{Cov}(y_{i,2j-1}, y_{i,2j})]/4h^2 \\ &= [2h\sigma_y^2 - 2h\rho_y\sigma_y^2]/4h^2 = (1-\rho_y)\sigma_y^2/2h \quad (7)\end{aligned}$$

since $\mathrm{Cov}(y_{ij}, y_{ik})=0$ if either $k \neq j$ or $k \neq j+1$ by Assumption 2. Similarly, we get

$$\begin{aligned}\mathrm{Cov}(\bar{\mathbf{c}}_i) &= \mathrm{Var}(\sum_{j=1}^{2h}\mathbf{c}_{ij})/4h^2 = [\sum_{j=1}^{2h}\mathrm{Var}(\mathbf{c}_{ij}) \\ &\quad + 2\sum_{j<k}\mathrm{Cov}(\mathbf{c}_{ij}, \mathbf{c}_{ik})]/4h^2 = \Sigma_c/2h \qquad (8)\end{aligned}$$

by Assumptions 4 and 5. Also the covariance between $\bar{y}_i$ and $\bar{c}_i$ is given by

$$\begin{aligned}\mathrm{Cov}(\bar{y}_i, \bar{\mathbf{c}}_i) &= \mathrm{Cov}(\sum_{j=1}^{2h} y_{ij}/2h, \sum_{j=1}^{2h}\mathbf{c}_{ik}/2h) \\ &= [\sum_{j=1}^{2h}\mathrm{Cov}(y_{ij}, \mathbf{c}_{ij}) \\ &\quad + \sum_{j\neq k}\mathrm{Cov}(y_{ij}, \mathbf{c}_{ik})]/4h^2 = \sigma'_{yc}/2h \qquad (9)\end{aligned}$$

by Assumptions 3. Thus, from (7)-(9), under the normality assumption of the response and control variates, the joint distribution of $\bar{y}_i$ and $\bar{c}_i$ is given by

$$\begin{bmatrix}\bar{y}_i \\ \bar{\mathbf{c}}_i\end{bmatrix} \sim N_{s+1}\left[\begin{bmatrix}\mathbf{x}'_i\beta \\ \mathbf{0}\end{bmatrix}, \frac{1}{2h}\begin{bmatrix}(1-\rho_y)\sigma_y^2 & \sigma'_{yc} \\ \sigma_{yc} & \Sigma_c\end{bmatrix}\right] \quad (10)$$

where x'_i is the ith row of X, and $x'_i\beta$ is the mean response at the ith design point.

The application of independent streams across the m design points allows that the $(s+1)$-variates simulation output, $(\bar{y}_i, \bar{c}'_i)$, at the ith design point is independent of $(\bar{y}_j, \bar{c}'_j)$ obtained at the different design point $(i \neq j)$. Therefore, under the joint normally assumption of the responses and control variates, we find the joint distribution of $\bar{y}$ and $\bar{C}$ as follows:

$$\begin{bmatrix}\bar{\mathbf{y}} \\ \mathrm{Vec}(\bar{\mathbf{C}})\end{bmatrix} \sim N_{m(s+1)}\left[\begin{bmatrix}\mathbf{X}\beta \\ \mathbf{0}_{ms}\end{bmatrix}, \ \Sigma\right]; \qquad (11)$$

where $\mathrm{Vec}(\bar{C})$ denotes the operation that the columns of $\bar{C}$ are stacked into a single ms-dimensional vector;

$$\Sigma = \frac{1}{2h}\begin{bmatrix}\sigma_y \mathrm{I}_m & \sigma'_{yc}\otimes \mathrm{I}_m \\ \sigma_{yc}\otimes \mathrm{I}_m & \Sigma_c \otimes \mathrm{I}_m\end{bmatrix}, \qquad (12)$$

where $\otimes$ denotes a Kronecker operation of two matrices. From Theorem 2.5.1 in Anderson,[1] the conditional variance of $\bar{y}$ given $\bar{C}$ is as follows:

$$\mathrm{Var}(\bar{y} \mid \bar{C}) = [\sigma_y I_m - (\sigma'_{yc} \otimes I_m)(\Sigma_c \otimes I_m)^{-1}$$
$$(\sigma_{yc} \otimes I_m)]/2h = [\sigma_y I_m - (\sigma'_{yc} \otimes I_m)(\Sigma_c^{-1} \otimes I_m)$$
$$(\sigma_{yc} \otimes I_m)]/2h = [\sigma_y I_m - (\sigma'_{yc}\Sigma_c^{-1} \otimes I_m)/2h$$
$$(\sigma_{yc} \otimes I_m)] = [\sigma_y I_m - (\sigma'_{yc}\Sigma_c^{-1}\sigma_{yc} \otimes I_m)]/2h$$
$$= [\sigma_y I_m - \sigma'_{yc}\Sigma_c^{-1}\sigma_{yc} I_m]/2h$$
$$= \sigma_y^2/2h\ (1 - \rho_y - R_{yc}^2) I_m, \qquad (13)$$

where Ryc is the multiple correlation coefficient between y_{ij} and c_{ij}. Equations (12) and (13) imply that the mean response vector $\bar{y}$ can be written as the linear model in (4) with an appropriate replacement of simulation output $\bar{c}$. Thus, the least wquares estimator of β_G in (4) is given by

$$\hat{\beta}_G \mid \bar{C} = (X'X)^{-1}X'[I_m - \bar{C}(\bar{C}'P\bar{C})^{-1}\bar{C}'P]\bar{y} \qquad (14)$$

(see (5)). Taking the operation of variance on (14) gives

$$\mathrm{Var}(\hat{\beta}_G \mid \bar{C}) = (X'X)^{-1}X'[I_m - \bar{C}(\bar{C}'P\bar{C})^{-1}\bar{C}'P]$$
$$\mathrm{Var}(\bar{y} \mid \bar{C})[I_m - P\bar{C}(\bar{C}'P\bar{C})^{-1}\bar{C}']X(X'X)^{-1},$$

which is developed into, by substituting for Var ($\bar{y}|\bar{C}$) with (13),

$$\mathrm{Var}(\hat{\beta}_G \mid \bar{C}) = \sigma_y^2/2h\ (1 - \rho_y - R_{yc}^2)(X'X)^{-1}X'$$
$$[I_m - \bar{C}(\bar{C}'P\bar{C})^{-1}\bar{C}'P][I_m - P\bar{C}(\bar{C}'P\bar{C})^{-1}\bar{C}']$$
$$X(X'X)^{-1} = \sigma_y^2/2h\ (1 - \rho_y - R_{yc}^2)[(X'X)^{-1} +$$
$$(X'X)^{-1}X'\bar{C}(\bar{C}'P\bar{C})^{-1}\bar{C}'X(X'X)^{-1}] \qquad (15)$$

since $X'P = PX = 0$. Since the least squares estimator $\hat{\beta}_G$ is an unbiased estimator conditionally on $\bar{C}$, the unconditional variance of $\hat{\beta}_G$ is given by

$$\mathrm{Var}(\hat{\beta}_G) = E[\mathrm{Var}(\hat{\beta}_G \mid \bar{C})] = \sigma_y^2/2h\ (1 - \rho_y - R_{yc}^2)$$
$$(m - p - 2/m - p - s - 2)(X'X)^{-1} \qquad (16)$$

(see the proof in Appendix). This equation indicates that this method reduces the variance of the estimator for $\beta_i (i = 0, 1, \cdots, p)$ by $(\rho_y + R^2_{yc})\sigma^2_y/2h$ and its loss factor is $(m-p-2)/(m-p-s-2)$ due to the estimation of α in (4), respectively, compared with the results obtained by independent streams across the $2h$ replications and m design points.

3.2. Combined Method II

In Section 3.1, we developed a combined method for the multipopulation experiments with the general linear model. Here, we consider a way of combined approach for a single population model to the multipopulation model in (4) where a design matrix X admits orthogonal blocking into two blocks. Basically, this extension to the multipopulation model involves combining in an additive manner the Schruben-Margolin correlation induction strategy and Combined Method I. Instead of directly applying Schruben-Margolin method across m design points, we first partition a set of the stochastic components in the model into two subsets of the non-control variate components and control variate components. Then we use correlation methods of common random numbers and antithetic variates partially through the non-control variate stochastic components in the model. Even though this correlation induction strategy may weaken the desired correlation of the responses at two design points, it allows the control variates to be observed independently at each design point. This last point is critical in order to achieve an additive effect from the two methods.

For the design matrix X admitting orthogonal blocking into two blocks, Schruben and Margolin [11] exploited the random number assignment rule which uses a combination of common random numbers and antithetic streams across m design points. Their assignment rule uses the same set of random number streams R for all m_1 design points in the first block, and uses the same set of antithetic random number streams R for all m_2 design points in the second block $(m = m_1 + m_2)$ within a replication.

Parallel to the work of Schruben and Margolin, [11] we partition m design points into two orthogonal blocks consisting of m_1 and m_2 design points respectively. For the ith design point in each block, the first set of streams R_{ij1} is selected according to the Schruben-Margolin assignment rule, and the second set of streams, R_{ij2}, are randomly selected through the $2h$ replications in the experiment $(j=1, 2, \cdots, 2h)$. On the other hand, for the $2h$ replications at the ith design point, the same assignment rule as in Section 3.1 is employed. For instance, within the first pair of replications, two different design points i k in the same block use (a) (R_{1l1}, R_{1l2}) and (R_{1l1}, R_{kl2}) respectively for the first replication; and (b) (R_{1l1}, R_{1l22}) and $(\bar{R}_{1l1}, R_{k22})$, respectively, for the second replication; where R_{1l1}, R_{1l2}, and R_{kl2} $(l=1, 2)$ are randomly selected, but R_{1l2} is antithetic to R_{1l1}. Table 2 presents the complete assignment of random number streams for the $2h$ replications at m design points: the first m_1 design points are in the first block, and the second m_2 design points are in the second block; R_{1j1} consists of $(g-s^*)$ random number streams used for the non-control stochastic components in the model $(j=1, 2, \cdots, m, j=1, 2, \cdots,)$; R_{1j2} consists of s^* random number streams used for the control variates in the model $(i=1, 2, \cdots, n, j=1, 2, \cdots, h)$; R_{1j1} is a set of randomly selected random number streams for the $(2j-1)$th replication $(j=1, 2, \cdots, h)$; R_{1j1} is a set of streams antithetic to $R_{1j1}(i=1, 2, \cdots, h)$; and R_{ij2} is a set of randomly selected random number streams for the jth replication at the ith design point $(i=1, 2, \cdots, m, j=1, 2, \cdots, 2h)$.

As before, let y_{ij} and c_{ij} be the response of interest and a vector of control variates, respectively, at the jth replication and the ith design point. We first specify the covariance matrix of the responses and control variates obtained by the assignment procedure described above. To this fend, we establish the following assumptions.

1. $Var(y_{ij})=\sigma^2_y$, for $i=1, 2, \cdots, m, j=1, 2, \cdots, 2h$ (homogeneity of response variances across design points and replicates).
2. $Cov(y_{ij}, y_{ik})=-\rho_y\sigma^2_y(\rho_y>0)$ for $i=1, 2, \cdots, m$ if $k=j+1 (j=1, 3, \cdots, 2h-1)$(homogeneity of response variances across design points and replicates pairs). Otherwise, $Cov(y_{ij}, y_{ik})=0$.
3. $Cov(y_{ij}, y_{kl})=\rho_{-}\sigma^2_y$ if two design points i and k are in the same block, and $l=j$; $Cov(y_{ij}, y_{kl})=\rho_{-}\sigma^2_y$ if two design points i and k are in the same block, and $l=j+1 (j=1, 3, \cdots, 2h-1)$; (homogeneity of induced correlations across design points: adopted from Schruben and Margolin)[11]. Otherwise, $Cov(y_{ij}, y_{kl})=0$.
4. $Cov(y_{ij}, y_{kl})=\rho_{-}\sigma^2_y$ if two design point i and

Table 2. Randon Number Assignment Rule of Combined Method Ⅱ

Design Point	Replication 1	2	$\cdots$	$2h-1$	$2h$
1	$y_{11}(R_{111}, R_{112})$	$y_{12}(\bar{R}_{111}, R_{122})$	$\cdots$	$y_{1,2h-1}(R_{1h1}, R_{1,2h-1,2})$	$y_{12h}(\bar{R}_{1h1}, R_{12h2})$
2	$y_{21}(R_{111}, R_{212})$	$y_{22}(\bar{R}_{111}, R_{222})$		$y_{2,2h-1}(R_{1h1}, R_{2,2h-1,2})$	$y_{22h}(\bar{R}_{1h1}, R_{22h2})$
	$\vdots$	$\vdots$		$\vdots$	$\vdots$
m_1	$y_{m_1 1}(R_{111}, R_{m_1 12})$	$y_{m_1 2}(\bar{R}_{111}, R_{m_1 22})$	$\cdots$	$y_{m_1,2h-1}(R_{1h1}, R_{m_1,2h-1,2})$	$y_{m_1 2h}(\bar{R}_{1h1}, R_{m_1 2h2})$
m_1+1	$y_{m_1+1,1}(\bar{R}_{111}, R_{m_1+1,12})$	$y_{m_1+1,2}(R_{111}, R_{m_1+1,22})$		$y_{m_1+1,2h-1}(\bar{R}_{1h1}, R_{m_1+1,2h-1,2})$	$y_{m_1+1,2h}(R_{1h1}, R_{m_1+1,2h2})$
	$\vdots$	$\vdots$		$\vdots$	$\vdots$
m	$y_{m1}(\bar{R}_{111}, R_{m12})$	$y_{m2}(R_{111}, R_{m22})$	$\cdots$	$y_{m,2h-1}(\bar{R}_{1h1}, R_{m,2h-1,2})$	$y_{m2h}(R_{1h1}, R_{m2h2})$

k are in two different blocks, and $j=1$; $Cov(y_{ij}, y_{kl}) = \rho_- \sigma^2_y$ if two design points i and k are in two different blocks, and $l=j+1 (j=1, 3, \cdots, 2h-1)$; (homogeneity of induced correlations across design points: adopted from Schruben and Margolin).[11] Otherwise, $Cov(y_{ij}, y_{kl}) = 0$.

5. ρ_+ and ρ_- are constant and $\rho_+ \geqq -\rho_- \geqq 0$ (standard statistical assumption and empirical simulation results: adopted from Schruben and Margolin [11]).

6. $Cov(y_{ij}, C_{kl}) = \sigma'_{yc}$ if $i=k$ and $j=l$ (homogeneity of control variates-response covariance across design points and replicates). Otherwise, $Cov(y_{ij}, C_{kl}) = 0'$.

7. $Cov(c_{ij}) = \Sigma_c$, for $i=1, 2, \cdots, m$ and $j=1, 2, \cdots, 2h$ (homogeneity of control variates covariance structure across design points and replicates).

8. $Cov(c_{ij}, c_{kl}) = O_{sxs}$ for $i \neq k$ and $j \neq l$ (independence of control variates across design points and replicates).

Assumptions 3, 4 and 5 are adopted from Schruben and Margolin,[11] and the other assumptions are the same as those in Section 3.1. Under these assumptions, we identify the conditional distribution of $\bar{y}$ given $\bar{C}$, where $\bar{y} = (\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_m)'$ and $\bar{C} = (\bar{c}_1, \bar{c}_2, \cdots, \bar{c}_m)'$.

Since this extension uses the same random number strategy as that considered in Section 3.1 at the ith design point, the variance of $\bar{y}_i$, the covariance of $\bar{c}_i$, and the covariance between $\bar{y}_i$ and $\bar{c}_i$ are, respectively, equivalent to those given in (7), (8), and (9). Thus, under the joint normality assumption of the responses and control variates, the joint distribution of $\bar{y}_i$ and $\bar{c}_i$ is same as that in (10).

Next we specify the covariance of the mean responses between two different design points. When y_i and y_k are the mean responses observed at two design points in the same block, we find (see Theorem 5.2 in Mood, Graybill and Boes [7]),

$$
\begin{aligned}
&Cov(\bar{y}_i, \bar{y}_k) \\
&= Cov\Big(\sum_{j=1}^{2h} y_{ij}/2h, \sum_{l=1}^{2h} y_{kl}/2h\Big) \\
&= \Big[\sum_{j=1}^{2h}\sum_{l=1}^{2h} Cov(y_{ij}, y_{kl})\Big] / 4h^2 \\
&= \sum_{j=1}^{h}\sum_{l=1}^{2h} [Cov(y_{i,2j-1}, y_{kl}) / 4h^2 + Cov(y_{i,2j}, y_{kl})] \\
&= \Big\{\sum_{j=1}^{h} [Cov(y_{i,2j-1}, y_{k,2j-1}) + Cov(y_{i,2j-1}, y_{k,2j}) \\
&\quad + \sum_{l \neq 2j-1,\, 2j} Cov(y_{i,2j-1}, y_{kl})] \\
&\quad + \Big[\sum_{j=1}^{h} Cov(y_{i,2j}, y_{k,2j}) \\
&\quad + Cov(y_{i,2j}, y_{k,2j-1}) + \sum_{l \neq 2j-1,\, 2j} Cov(y_{i,2j}, y_{kl})\Big]\Big\} / 4h^2 \\
&= \Big[\sum_{j=1}^{2h} Cov(y_{ij}, y_{kj}) + \sum_{i=1}^{h} \{Cov(y_{i,2j-1}, y_{k,2j}) \\
&\quad + Cov(y_{i,2j}, y_{k,2j-1})\}\Big] / 4h^2 \\
&= [2h\rho_+\sigma_y^2 + 2h\rho_-\sigma_y^2] / 4h^2 = (\rho_+ + \rho_-)\sigma_y^2 / 2h
\end{aligned}
\tag{17}
$$

by Assumption 3. Also, for the mean response $\bar{y}_i$ and $\bar{y}_k$ in two different blocks, we have

$$
Cov(\bar{y}_i, \bar{y}_k) = (\rho_+ + \rho_-)\sigma_y^2 / 2h \tag{18}
$$

Next, the covariance between $\bar{y}_i$ and $\bar{C}_k$ for $i \neq k$ is given by

$$
\begin{aligned}
Cov(\bar{y}_i, \bar{c}_k) &= Cov\Big(\sum_{j=1}^{2h} y_{ij}/2h, \sum_{l=1}^{2h} c_{kl}/2h\Big) \\
&= \Big[\sum_{j=1}^{2h}\sum_{l=1}^{2h} Cov(y_{ij}, c_{kl})\Big] / 4h^2 = 0.
\end{aligned}
\tag{19}
$$

by Assumption 6. Finally, we have

$$\mathrm{Cov}(\bar{c}_i, \bar{c}_k) = \mathrm{Cov}\Big(\sum_{j=1}^{2h} c_{ij}/2h\ ,\ \sum_{l=1}^{2h} c_{kl}/2h\Big)$$

$$= \Big[\sum_{j=1}^{2h}\sum_{l=1}^{2h} \mathrm{Cov}(c_{ij}, c_{kl})\Big]/4h^2 = 0_{s\times s} \qquad (20)$$

by Assumption 8.

Under the joint normality assumption of the response and control variates, from equations (7)-(9) in Section 3.1, and (17)-(20), we find the joint distribution of $\bar{y}$ and $\bar{C}$ as follows:

$$\begin{bmatrix} \bar{\mathbf{y}} \\ \mathrm{Vec}(\bar{\mathbf{C}}) \end{bmatrix} \sim N_{m(s+1)}\left[\begin{bmatrix} \mathbf{X}\beta \\ \mathbf{0}_{ms} \end{bmatrix},\ \Sigma\right]; \qquad (21)$$

where Vec(C) denotes the operation that the columns of $\bar{C}$ are stacked into a single *ms*-dimensional vector;

$$\Sigma = \begin{bmatrix} \Sigma_y & \sigma'_{yc}\otimes \mathbf{I}_m \\ \sigma_{yc}\otimes \mathbf{I}_m & \Sigma_c \otimes \mathbf{I}_m \end{bmatrix} \Big/ 2h \qquad (22)$$

where $\otimes$ denotes a Kronecker operation of two matrices; and

$$\Sigma_y = \sigma_y^2 \begin{bmatrix} 1-\rho_y & \rho_+ + \rho_- & & \rho_+ + \rho_- & \rho_+ + \rho_- & \rho_+ + \rho_- & & \rho_+ + \rho_- \\ \rho_+ + \rho_- & 1-\rho_y & \vdots & \rho_+ + \rho_- & \rho_+ + \rho_- & \rho_+ + \rho_- & \vdots & \rho_+ + \rho_- \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho_+ + \rho_- & \rho_+ + \rho_- & \vdots & 1-\rho_y & \rho_+ + \rho_- & \rho_+ + \rho_- & \vdots & \rho_+ + \rho_- \\ \rho_+ + \rho_- & \rho_+ + \rho_- & \vdots & \rho_+ + \rho_- & 1-\rho_y & \rho_+ + \rho_- & \vdots & \rho_+ + \rho_- \\ \rho_+ + \rho_- & \rho_+ + \rho_- & \vdots & \rho_+ + \rho_- & \rho_+ + \rho_- & 1-\rho_y & \vdots & \rho_+ + \rho_- \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \rho_+ + \rho_- & \rho_+ + \rho_- & & \rho_+ + \rho_- & \rho_+ + \rho_- & \rho_+ + \rho_- & & 1-\rho_y \end{bmatrix}$$

$$= [(1-\rho_y-\rho_+-\rho_-)\mathbf{I}_m + (\rho_+ + \rho_-)\mathbf{1}_m\mathbf{1}'_m]\ \sigma_y^2/2h \qquad (23)$$

From Theorem 2.5.1 in Anderson [1], the conditional variance of $\bar{y}$ given $\bar{C}$ is as follows:

$$\mathrm{Var}(\bar{\mathbf{y}} \mid \bar{\mathbf{C}}) = [\Sigma_y - (\sigma'_{yc}\otimes\mathbf{I}_m)(\Sigma_c\otimes\mathbf{I}_m)^{-1}(\sigma_{yc}\otimes\mathbf{I}_m)]/2h$$

$$= [\Sigma_y - (\sigma'_{yc}\otimes\mathbf{I}_m)(\Sigma_c^{-1}\otimes\mathbf{I}_m)(\sigma_{yc}\otimes\mathbf{I}_m)]/2h$$

$$= [\Sigma_y - (\sigma'_{yc}\Sigma_c^{-1}\otimes\mathbf{I}_m)(\sigma_{yc}\otimes\mathbf{I}_m)]/2h$$

$$= [\Sigma_y - (\sigma'_{yc}\Sigma_c^{-1}\sigma_{yc}\otimes\mathbf{I}_m)]/2h$$

$$= [\Sigma_y - \sigma'_{yc}\Sigma_c^{-1}\sigma_{yc}\mathbf{I}_m]/2h$$

$$= [(1-\rho_y-\rho_+-\rho_-)\mathbf{I}_m + (\rho_+ + \rho_-)\mathbf{1}\mathbf{1}'$$

$$-\sigma_y^{-2}\sigma'_{yc}\Sigma_c^{-1}\sigma_{yc}\mathbf{I}_m]\ \sigma_y^2/2h$$

$$= [(1-\rho_y-\rho_+-\rho_- - R_{yc}^2)\mathbf{I}_m$$

$$+ (\rho_+ + \rho_-)\mathbf{1}\mathbf{1}']\ \sigma_y^2/2h$$

$$= \gamma\mathbf{I}_m + \delta\mathbf{1}\mathbf{1}', \qquad (24)$$

where R_{yc} is the multiple correlation coefficient between y_{ij} and c_{ij}. A sufficient condition for the equivalence of OLS and WLS estimators is that the dispersion matrix in (24) is representable as

$$\mathrm{Var}(\bar{\mathbf{y}} \mid \bar{\mathbf{C}}) = \mathrm{X}[\,\mathrm{I}\,]\mathrm{X}' + z\theta z' + \sigma^2\mathrm{I}_m, \qquad (25)$$

where z is an $(m\times 1)$ vector such that $z'X=0$, and [I], θ, and σ^2 are arbitrary (see equation (63) in Rao [10], and equation (3.6) in Schruben and Margolin [11]). Clearly, the covariance matrix in (24) is of the form in (25) since

$$\mathrm{Var}(\bar{\mathbf{y}} \mid \bar{\mathbf{C}}) = (1-\rho_y-\rho_+-\rho_- - R_{yc}^2)\sigma_y^2\mathbf{I}_m/\ 2h$$

$$+ (\rho_+ + \rho_-)\sigma_y^2\mathbf{X}\mathbf{G}_{p+1}\mathbf{X}'/\ 2h \qquad (26)$$

where G_{p+1} is a $((p+1)\times(p+1))$ matrix whose first row and first column entry is 1 with all other entries 0. The OLS estimator of β_G in (4) is given by

$$\hat{\beta}_G \mid \bar{\mathbf{C}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'[\mathbf{I}_m - \bar{\mathbf{C}}(\bar{\mathbf{C}}'\mathbf{P}\bar{\mathbf{C}})^{-1}\bar{\mathbf{C}}'\mathbf{P}]\bar{\mathbf{y}}. \qquad (27)$$

Taking the operation of variance on (27) gives

$$\mathrm{Var}(\hat{\beta}_G \mid \bar{C}) = (X'X)^{-1}X'[I_m - \bar{C}(\bar{C}'P\bar{C})^{-1}\bar{C}'P] \mathrm{Var}(\bar{y} \mid \bar{C})[I_m - P\bar{C}(\bar{C}'P\bar{C})^{-1}\bar{C}']X(X'X)^{-1},$$

which is developed into, by substituting for Var $(\bar{y}|\bar{C})$ with (24),

$$\mathrm{Var}(\hat{\beta}_G \mid \bar{C}) = \gamma(X'X)^{-1}X'[I_m - \bar{C}(\bar{C}'P\bar{C})^{-1}\bar{C}'P][I_m - P\bar{C}(\bar{C}'P\bar{C})^{-1}\bar{C}']X(X'X)^{-1} + \delta(X'X)^{-1}X'[I_m - \bar{C}(\bar{C}'P\bar{C})^{-1}\bar{C}'P]I_m I'_m[I_m - P\bar{C}(\bar{C}'P\bar{C})^{-1}\bar{C}']X(X'X)^{-1}. \quad (28)$$

The first term in this equation reduces to

$$\gamma(X'X)^{-1}X'[I_m - \bar{C}(\bar{C}'P\bar{C})^{-1}\bar{C}'P][I_m - P\bar{C}(\bar{C}'P\bar{C})^{-1}\bar{C}']X(X'X)^{-1} = \gamma(X'X)^{-1} + \gamma(X'X)^{-1}X'\bar{C}(\bar{C}'P\bar{C})^{-1}\bar{C}'X(X'X)^{-1} \quad (29)$$

since $X'P=PX=0$. After some computations, the second term is equivalent to

$$\delta(X'X)^{-1}X'[I_m - \bar{C}(\bar{C}'P\bar{C})^{-1}\bar{C}'P]I_m I'_m[I_m - P\bar{C}(\bar{C}'P\bar{C})^{-1}\bar{C}']X(X'X)^{-1} = \delta G_{p+1} \quad (30)$$

(see the proof in Appendix). Substituting (29) and (30) into (28), we find the conditional variance of $\hat{\beta}_G$, given $\bar{C}$, as follows:

$$\mathrm{Var}(\hat{\beta}_G \mid \bar{C}) = \gamma[(X'X)^{-1} + (X'X)^{-1}X'\bar{C}(\bar{C}'P\bar{C})^{-1}\bar{C}'X(X'X)^{-1}] + \delta G_{p+1} \quad (31)$$

Since the least squares estimator $\hat{\beta}_G$ is an unbiased estimator conditionally on $\bar{C}$, the unconditional variance of $\hat{\beta}_G$ is given by

$$\mathrm{Var}(\hat{\beta}_G) = E[\mathrm{Var}(\hat{\beta}_G \mid \bar{C})] = \gamma(m-p-2)/(m-p-s-2)(X'X)^{-1} + \delta G_{p+1} \quad (32)$$

(see the proof in Appendix A).

4. Example

We applied Combined Method Ⅱ developed in Section 3.2 to a hospital resource allocation model [11] for an illustration of implementation. We present the experimental performance of this variance reduction technique.

4.1. Description of System and Model

Figure 1 shows the operation of the hospital unit in terms of patient paths and types of resource (see Figure A in Schruben and Margolin [11]). The hospital unit consists of three types of resources that are devoted to specialized care: intensive care, coronary care and intermediate care. Patients arrive at the hospital unit according to a poisson process with an arrival rate of 3.3 per day. Upon entering the hospital, 75% of the patients need intensive care, and 25% need coronary care. The service time distribution at intensive care is lognormal with mean 3.8 days and standard deviation 3.5 days, that of coronary care is lognormal with mean 3.8 days and standard deviation 1.6 days. After intensive care, 27% of the patients leave the hospital and 73% go to the intermediate care unit. Also, completing the coronary care, 20% of the patients leave the system and 80% go to the intermediate care unit. Intermediate care stay for intensive care patients is distributed lognormally with mean 15.0 days and standard deviation 7.0 days. Finally, the length of intermediate care for coronary patients is distributed lognormally with mean 17.0 days and standard deviation 3.0 days. When the patients request admission to special care units which are unavailable, they can not be accommodated and balk from the system.

The hospital administration now considers construction of a new facility to provide better service to the patients. The administration's decision is

Table 3. Experimental Design Points in 2^3 Factorial Design

	Experimental Design Point	Number of Beds (Intensive)	Number of Beds (Coronary)	Number or Beds (Intermediate)
Block 1	1	13(-1)	4(-1)	15(-1)
	2	13(-1)	6(-1)	17(1)
	3	15(1)	4(1)	17(1)
	4	15(-1)	6(1)	15(-1)
Block 2	5	13(-1)	4(-1)	17(1)
	6	13(-1)	6(1)	15(-1)
	7	15(1)	4(-1)	15(-1)
	8	15(1)	6(1)	17(1)

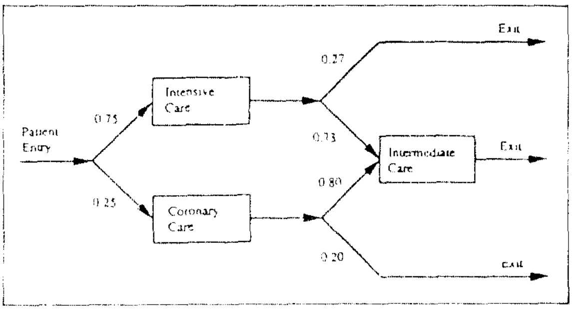

Figure 1. Hospital Resource Allocation Model

complicated by conflicting interest of several groups because no one knows how the numbers of each type of bed will affect the frequency with which the patients can not be accommodated. To help resolve this conflict, a statistically designed simulation experiment is conducted.

Schruben and Margolin [11] illustrated this problem to investigate the simulation efficiency of their correlation induction strategy. For estimating the effect of the number of beds of each type to the failure rate of the patients, they implemented a 2^3 factorial design: three factor variables (three types of beds) having two levels for each factor. The experimental conditions for the eight design points in the 2^3 factorial design are given in Table 3. They also proposed a linear model which includes an overall mean and all main effects and pairwise interactions. Their simulation results showed that two factor interaction effects are negligible. Based on these results, in the application of Combined Method Ⅱ to this model, we consider a linear model consisting only of the overall mean response and all main effects.

In simulating this model, we used the single standardized control variate of interarrival time of the patients to the system:

$$c_{ij} = (a_{ij}(t))^{-1/2} \sum_{j=1}^{a_{ij}(t)} (s_{ij} - \mu_k)/\sigma_k \quad ; \qquad (33)$$

where μ_k and σ_k are the known mean and standard deviation of the service time at station k; $s_{ij}(j=1, 2, \cdots, a_i(t))$ are the random observations of the interarrival times of the patients at the ith design point and jth replication; and $a_{ij}(t)$ is the number of observations of s_{ij} during the simulation the simulation time $(0, t)$ (see Wilson and Pritsker [16]). The interarrival times of the patients to the system would be independently observed at each level of the factor variables (service times at the three hospital units) since we use different number streams for driving the arrival process of the patients to the system. Thus, we can assume that this control variate is independent of the three factor variables. Including this control variate to the linear model of the factor variables, we have

$$\bar{y}_i = \beta_0 + \sum_{j=1}^{3} \beta_j x_{ij} + \bar{c}_i \alpha + \varepsilon_i^*, \quad i = 1, 2, \ldots, 8, \tag{34}$$

where $\bar{y}_i$ is the average failure rate (the response of interest) at the ith design point; β_0 is the overall mean; β_j is the main effect of the jth factor variable (number of the specialized care beds); x_{ij} is 1 (-1) if the jth factor is at the high (low) level for a design point i (by a reparemeterization of the factor variables); $\bar{c}_i$ is the mean control variate at the ith design point; and ε^*_i is the error for the ith observation. Clearly, the (8×4) design matrix $X=(x_{ij})$ given in (34) admits orthogonal blocking into two blocks. We partitioned the eight design points of the design matrix X into two blocks: the first block includes the design points 1-4, and the second block includes the design points 5-8 (see Table 3).

A single replication at each design point used eight separate random number streams for driving eight stochastic model components to the assignment rule in Table 2 (note that we use one control variate of interarrival time). Within a paired replications at each design point, (a) the first replicate randomly selected eight random number streams, but (b) the second replicate randomly selected random number stream 1 (used for driving the interarrival process), and used the other streams antithetic to those used in the first replication for driving the non-control stochastic variates. Across the design points, this method used independent random number streams for generating the interarrival time process (control variate), but employed the Schruben-Margolin random number assignment rule for driving the non-control variates stochastic model components.

We coded this model in SLAM Ⅱ and conducted 200 replications at each design point. One replication consists of simulating the system for 1500 days. To reduce the initial bias, we collected the necessary statistics after a warm-up period of length 300 time units.

4.2. Experimental Results

We computed the performance statistics of the D-value (determinant) of the estimated covariance matrix of the parameters, and the variances of the estimators for the parameters obtained by this method. Let $2h$ be the number of replications at each design point and $y_j=(y_{1j}, \cdots, y_{8j})'$ be the response vector of the eight design points for the jth replication. Also let $c_j=(c_{1j}, c_{2j}, \cdots, c_{8j})'$ be the vector of control variates corresponding to y_j. The adjusted mean responses for the eight design points are given by

$$\bar{\mathbf{y}}(\hat{\alpha}) = \bar{\mathbf{y}} - \hat{\alpha}\bar{\mathbf{c}} \tag{35}$$

where $\bar{y}=\sum_{j=1}^{2h} y_j/2h$ is the mean response vector at the eight design points, $\bar{c}=(\bar{c}_1, \bar{c}_2, \cdots, \bar{c}_8)'$ is the mean vector of control variates at the eight design points, and $\hat{\alpha}$ is the least squares estimator of the linear model in (34), given by

$$\hat{\alpha} = (\bar{\mathbf{c}}'\mathbf{P}\bar{\mathbf{c}})^{-1}\bar{\mathbf{c}}'\mathbf{P}\bar{\mathbf{y}}. \tag{36}$$

Thus, the covariance matrix of the adjusted responses at the eight design points is estimated by

$$\mathbf{S}_{y(\hat{\alpha})} = \sum_{j=1}^{2h} (\mathbf{y}_j(\hat{\alpha}) - \bar{\mathbf{y}}(\hat{\alpha}))(\mathbf{y}_j(\hat{\alpha}) - \bar{\mathbf{y}}(\hat{\alpha}))'/(2h-1) \tag{37}$$

where $y_j(\hat{\alpha})=y_j-\hat{\alpha}c_j$. Given the control variates, the sample covariance matrix of $\hat{\beta}_G$ is given by

$$Cov(\hat{\beta}_C) = (X'X)^{-1}X'S_{y(\hat{\alpha})}X(X'X)^{-1} \quad (38)$$

Using the computational procedures mentioned above, we obtained appropriate statistics. We summarize the simulation results: (a) Table 4 gives the sample correlation matrix of the responses at the eight design points, (b) Table 5 provides the estimators for the parameters β, and (c) Table 6 presents the covariance matrix of the estimators for the parameters, and its D-values.

4.3. Inferences

The correlation coefficients between two adjusted responses either in the same block or different blocks are in the range from 0.58 to 0.74. In comparing to the induced correlation matrix of the Schruben-Margolin method (see the simulation results in Tew and Wilson [14] and Tew [13]), it seems more difficult to obtain the assumed correlation matrix structure (equal correlation between the two responses) in applying this method. This result indicates that the assumptions on the equal correlations between the two responses in either the same block or different blocks (Assumptions 3 and 4 in Section 3.2) need the analytical and empirical validation although similar assumptions of Schruben and Margolin [11] are generally accepted. We conjecture that this is due to the use of independent random number streams for driving the control variate across design points wheih reduces the synchronization effect of random number streams in applying this method. However, this method yields positive correlations between any two controlled responses with values not much less than those induced by the Schruben-Margolin method for the responses in the same block.

5. Conclusions

In reducing the variances of the estimators for the parameters, Combined Methods I and II focus on reduction of the variances of the mean responses at each design point by applying antithetic variates and control variates simultaneously. Combined Method II additionally tries to take advantage of the Schruben-Margolin method by inducing correlations between any two responses in the design after the control variate effect has been accounted for. In applying the Schruben-Margolin method, the magnitude of the correlation coefficient between two responses in the same block is critical to the efficiency of this method in reducing the variances of the estimators for the main (interaction) effects of the factor variables. When synchronization of the rendom number streams is easily achieved, the Schruben-Margolin method may show good per-

Table 4. Correlation Matrix of Adjusted Responses

	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
y_1	1.000	0.634	0.584	0.624	0.617	0.642	0.611	0.601
y_2	0.634	1.000	0.709	0.657	0.637	0.741	0.706	0.730
y_3	0.584	0.709	1.000	0.680	0.567	0.672	0.699	0.667
y_4	0.624	0.657	0.680	1.000	0.617	0.619	0.597	0.644
y_5	0.617	0.637	0.567	0.617	1.000	0.718	0.664	0.729
y_6	0.642	0.741	0.672	0.619	0.718	1.000	0.734	0.699
y_7	0.611	0.706	0.699	0.597	0.664	0.734	1.000	0.698
y_8	0.601	0.730	0.667	0.644	0.729	0.699	0.698	1.000

Table 5. Estimators for Model Parameters

Parameter	Combined Method II
β_0	45.588
β_1	-0.309
β_2	-0.384
β_3	1.809

Table 6. Covariance Matrix of Estimators for Model Parameters

Combined Method Ⅱ : D-Value $= 1.922\times10^{-7}$

	β_0	β_1	β_2	β_3
β_0	0.3255563	0.0080371	0.0063543	0.0038048
β_1	0.0080371	0.0210071	0.0017162	0.0030887
β_2	0.0063543	0.0017162	0.0170880	0.0040829
β_3	0.0038048	0.0030887	0.0040829	0.0181698

formance in estimating the model parameters. For the case that an effective set of control variates can be identified and synchronization of the random number streams is difficult to achieve in the model, Combined Methods may yield good results.

Appendix

Proof of Equation(16):

From Theorem 2.4.3 in Anderson [1], the marginal distribution of c_i in (10) is given by

$$\bar{c}_i \sim N_s(0,\ \Sigma_c/2h). \qquad (A1)$$

Since c_i ($i=1,\ 2,\ \cdots,\ m$) are independent by Assumption 5 in Section 3.1, we have

$$\bar{C} \sim N_{m,\ s}(O,\ \Sigma_c/2h,\ I_m). \qquad (A2)$$

and

$$\bar{C}'P\bar{C} \sim W_s(m-p-1,\ \Sigma_c/2h). \qquad (A3)$$

Since $\bar{C}(\Sigma_c/2h)^{-1/2} \sim N_{m,s}(O,\ I_s,\ I_m)$ from (A_2), we have

$$E[\bar{C}(\Sigma/2h)^{-1}\bar{C}'] = sI_m \qquad (A4)$$

(see Theorem 17.6a in Arnold [2]). Also from Theorem 17.15d in Arnold [2],

$$E[(\bar{C}'P\bar{C})^{-1}] = [2h/(m-p-s-2)]\Sigma_c^{-1}$$
$$\text{if } m > p-s-2. \qquad (A5)$$

Since the matrix P indicates that P is a symmetric and idempotent matrix with rank $(m-p-1)$, P is a positive semi-definite matrix(see Theorem 71.7.1 in Graybill [4]). In such a case, $\bar{C}X$ and $\bar{C}'P\bar{C}$ are independent since $PX=0$(see Theorem 4.5.1 in Graybill [4]). Therefore, by (A5),

$$E[X'\bar{C}(\bar{C}'P\bar{C})^{-1}\bar{C}'X] = E[X'\bar{C}E[(\bar{C}'P\bar{C})^{-1}]\bar{C}'X]$$
$$= [1/(m-p-s-2)]\ E[X'\bar{C}(\Sigma_c/2h)^{-1}\bar{C}'X], \qquad (A6)$$

which further reduces to

$$E[X'\bar{C}(\bar{C}'P\bar{C})^{-1}\bar{C}'X] = [1/(m-p-s-2)]\ X'$$
$$(sI_m)X = [s/(m-p-s-2)]X'X \qquad (A7)$$

by (A4). Therefore, taking the operation of expectation on (5) finally yields

$$Var(\hat{\beta}_G) = E[Var(\hat{\beta}_G \mid \bar{C})] = \sigma_y^2/2h\,(1-\rho_y-R_{yc}^2)$$
$$(X'X)^{-1}[1+s/(m-p-s-2)(X'X)(X'X)^{-1}]$$
$$= \sigma_y^2/2h\,(1-\rho_y-R_{yc}^2)$$
$$[(m-p-2)/(m-p-s-2)](X'X)^{-1}, \qquad (A8)$$

which is equivalent to(16)

Proof of Equation (30):

Note that 1_m is the first column of X and PX=0. Therefore, we have

$$P1_m = 1_m P = 0 \qquad (A9)$$

since PX=X'P=0. Developing the second term in (38) gives

$$\begin{aligned}&\delta(X'X)^{-1}X'[I_m - \bar{C}(\bar{C}'P\bar{C})^{-1}\bar{C}'P]1_m 1'_m[I_m - \\ &P\bar{C}(\bar{C}'P\bar{C})^{-1}\bar{C}']X(X'X)^{-1} = \delta(X'X)^{-1}X'[1_m 1'_m \\ &- \bar{C}(\bar{C}'P\bar{C})^{-1}\bar{C}'P1_m 1'_m - 1_m 1'_m P\bar{C}(\bar{C}'P\bar{C})^{-1}\bar{C}' \\ &+ \bar{C}(\bar{C}'P\bar{C})^{-1}\bar{C}'P1_m 1'_m P\bar{C}(\bar{C}'P\bar{C})^{-1}\bar{C}']X(X'X)^{-1} \\ &= \delta(X'X)^{-1}X'1_m 1'_m X(X'X)^{-1} \qquad (A10)\end{aligned}$$

by (A9). Since X is orthogonal, $X'1_m=(m, 0, \cdots, 0)'$, which implies

$$X'1_m 1'X_m = m^2 G_{p+1}, \qquad (A11)$$

where G_{p+1} is as defined in (26). Thus we have

$$\begin{aligned}(X'X)^{-1}X'1'_m 1_m X(X'X)^{-1} &= m^{-1}m^2 G_{p+1} m^{-1} \\ &= G_{p+1}. \qquad (A12)\end{aligned}$$

Substitution (A12) into (A10) finally yields (30).

Proof of Equation (32):

From equations (21) and (22), the marginal distribution of $\bar{C}$ in (21) is same as given in (A2). Therefore, using the same procedures in (A3)-(A7), we find

$$E[X'\bar{C}(\bar{C}'P\bar{C})^{-1}\bar{C}'X] = [s/(m-p-s-2)]X'X. \qquad (A13)$$

Thus, taking the operation of expectation on (31) yields

$$\begin{aligned}Var(\hat{\beta}_G) &= E[Var(\hat{\beta}_G \mid \bar{C})] = \gamma(X'X)^{-1} \\ &[1 + s/(m-p-s-2)(X'X)(X'X)^{-1}] + \delta G_{p+1} \\ &= \gamma(m-p-2)/(m-p-s-2)(X'X)^{-1} \\ &+ \delta G_{p+1},\end{aligned}$$

which is equivalent to (32).

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