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STABILIZABILITY AND CONTROL PROPERTIES FOR AN EVOLUTION EQUATION

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1. Introduction

The main results of this paper are to derive an estimate for A^{is} where $s \in \mathcal{R}$ and A is the realization of an elliptic operators with Dirichlet boundary condition and concerned with some controll results of the following equation

$$(1.1) \qquad \begin{cases} u'(t) + Au(t) = \Phi f(t). \quad 0 \le t \le T \\ u(0) = 0 \end{cases}$$

where $-A$ is the infinitesimal generator of an analytic semigroup in a complex Banach space X. The space X is colled δ-convex if there exists a real valued function δ on $X \times X$ having the properties

$$\delta(x, \cdot) \text{ is convex for each } x \in X$$
$$\delta(x,y) = \delta(y,x)$$
$$\delta(x,y) \le |x+y| \quad if \quad |x| \le 1 \le |y|$$

and $\quad \delta(0,0) > 0.$

For example, the sobolev space $L^p(\Omega)$ and l^p are δ-convex for $1 < p < \infty$, while $L^1(\mathcal{R},\mathcal{R})$ and l^p are not.

In [9], R. Seely established a similar results by estimating for the L^p norms of the complex power AB^z where AB is an elliptic operator A whose domain is defined by well posed boundary condition $Bu=0$.

From the R. Seely's estimate of AB^{x+iy} for $x < 0$,we now can derive the estimate of A^{is} for $s \in \mathcal{R}$ where A is generlized second order elliptic operator and for any $f \in L^r(0,T;L^p(\Omega))$ and $1 < r < \infty$ the equation (1.1) has a solution $u \in W^{1,r}(0,T;W_0^{1,p}) \cap L^r(0,T;\mathcal{D}(A))$

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where $W^{m,p}(\Omega)$ is the set of all functions whose derivative up to degree m in distribution sense belong to $L^p(\Omega)$ and the closure of $C_0^m(\Omega)$ in $W_p^m(\Omega)$ is denoted by $W^{m,p}(\Omega)$.

In section 3, we consider a necessary and sufficient condition for controllability for the general evolution equation in reflexive Banach space X. The criteria for controllability can be stated in terms of A^*, the adjoint operator of the infinitesimal generator A.

We also derive to the relation between stabilizability of solution and the controllability for the equation (1.1).

In section 4, we give the example of retarded system.In this case many author's have discussed these concepts for retarded and neutral systems [3,5,7].

In this note, we deal with also the stabilizability of retarded case, and the relation between stabilizable and controllable.

2. The group property of A^{is}

Let Ω be a bounded domain in $\mathcal{R}^n$ with smooth boundary ∂G.

If we set

$$(2.1)\qquad \mathcal{A} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial}{\partial x_j}) + \sum_{i=1}^{n} b_i\frac{\partial}{\partial x_i} + c$$

where $a_{ij} = a_{ji} \in C^1(\Omega)$ and $a_{ij}(x)$ is positive definite uniformly in $\bar{\Omega}$, $b_i \in C^1(\bar{\Omega})$ and $c \in L^\infty(\Omega)$, then the dual operator $\mathcal{A}'$ of $\mathcal{A}$ is

$$(2.2)\qquad \mathcal{A}' = -\sum_{i,j=1}^{n} a_j\frac{\partial^2}{\partial x_i\partial x_j} - \sum_{i=1}^{n}\frac{\partial}{\partial x_i}(b_i\cdot) + \bar{c}$$

Let A_p be $L^p(\Omega)$-realization with boundary value problem, that is,

(2.3)

$$\mathcal{D}(A_p) = \begin{cases} W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega) & = \{u\in W^{2,p}(\Omega) : u|_{\partial\Omega}=0\} \\ & ,if\ \ 1<p<\infty \\ \{u;\, u\in W^{1,q}(\Omega), & \mathcal{A}u\in L^1(\Omega)\ \ for\, 1\le q\le \frac{n}{n-1}\}, \\ & ,if\ \ p=1 \end{cases}$$

and $A_p u = \mathcal{A}u$ for $u \in \mathcal{D}(A_p)$, then $-A_p$ generates an analytic semi-groups in $L^p(\Omega)$.

Similarly, we define

$$\mathcal{D}(A_{p'} = W^{2,p'}(\Omega) \cap W_0^{1,P'}(\Omega), \quad \frac{1}{p} + \frac{1}{p'} = 1$$
$$A_{p'}u = \mathcal{A}'u$$

for $u \in \mathcal{D}(A_{p'})$, then $-A_{p'}$ also generates an analytic semigroup in $L^{p'}(\Omega)$.

We consider the regular Dirichlet boundary problem,now.

An elliptic Dirichlet boundary value problem is regular problem if its system has the smoothness assumptions on the domain and the coefficient introduced above all.

For the sake of simplicity,we assume that $0 \in \rho(A_p)$ and $0 \in \rho(A_p')$. Since $-A_p$ generates an analytic semigroup, there exists $w \in (0, \frac{\pi}{2})$ such that

$$\Sigma = \{\lambda : w \leq \arg \lambda \leq 2\pi - r\} \subset \rho(A_p). \tag{2.4}$$

We can define

$$(A_p)^z = \frac{1}{2\pi i}\int_\Gamma \lambda^z(\lambda - A_p)^{-1}\, d\lambda, \quad Re\, z < 0$$

where the path Γ runs in the resolvent set of A_p from $\infty e^{i\theta}$ to $\infty e^{-i\theta}$, $w < \theta < \pi$.

In [(a)], it is established that there exists a constant c such that $\| A_p^{x+iy} \|_p \leq ce^{\gamma(y)}$, $x < 0$ where c depands on A_p, ρ and γ is the constant in (2.4).

LEMMA 2.1. *A_p^{is}, $s \in \mathcal{R}$ is a bounded if and only if $A_p^{-\varepsilon+is}$ is a bounded for any $\varepsilon > 0$.*

Proof. Let A^{is} be a bounded for any $s \in \mathcal{R}$. It is known that there exists a constant $M > 0$ such that

$$\| A^{-\epsilon} \| \leq M$$

for $0 \leq \varepsilon \leq 1$.

For any $x \in \mathcal{D}(A)$ and $\alpha > 0$.

$$A^{-\alpha}x - x = A^{-\alpha}A^{-1}Ax - x = (A^{-\alpha-1} - A^{-1})Ax \longrightarrow 0$$

as $\alpha \longrightarrow 1$ $(\alpha \downarrow 1)$.

This shows that $A^{-\alpha} \longrightarrow I$ strongly as $\alpha \longrightarrow 0$, therefore it follows that for any $\varepsilon > 0$.

$$\| A^{-\varepsilon} \| \leq M$$

for the sufficiently large constant $M > 0$.

We have

$$\| A^{-\varepsilon+is} \| = \| A^{-\varepsilon}A^{is} \| \leq M \| A^{is} \|$$

where $\| A^{-\varepsilon} \| \leq M$

Hence, $A^{-\varepsilon+is}$ is a bounded for each $\varepsilon > 0$.

Conversely, for any $\varepsilon > 0$, let $\| A^{-\varepsilon+is} \|$ be bounded, that is $\| A^{-\varepsilon+is} \| \leq c$. For any $x \in \mathcal{D}(A^{is})$

$$A^{-\varepsilon+is}x = A^{-\varepsilon}A^{is}x \longrightarrow A^{is}x$$

as $\varepsilon \longrightarrow 0$. For $x \in L^p$ *and* $y \in \mathcal{D}(A^{is})$

$$\begin{aligned} \| A^{-\varepsilon'+is}x - A^{-\varepsilon+is}x \| &\leq \| A^{-\varepsilon'+is}(x-y) \| \\ &+ \| A^{-\varepsilon'+is}y - A^{-\varepsilon+is}y \| + \| A^{-\varepsilon+is}(y-x) \| \\ &\leq 2c \| x - y \| + \| A^{-\varepsilon'+is}y - A^{-\varepsilon+is}y \| . \end{aligned}$$

Hence the sequence $\{ A^{-\varepsilon+is}x \}$ is Cauchy sequence, there is $z \in L^p$ such that $A^{-\varepsilon+is}x \longrightarrow z$ *as* $\varepsilon \longrightarrow 0$.

Since $A^{-\varepsilon+is}x = A^{is}A^{-\varepsilon}x \longrightarrow z$ and $A^{-\varepsilon} \longrightarrow x$ it follows from closedness of A that $x \in \mathcal{D}(A^{is})$. and $A^{is}x = z$. Therefore A^{is} is a bounded in view of closed graph theorem.

LEMMA 2.2. *For any $s \in \mathcal{R}$, there exist a constant c such that $\| A^{is} \|_p \leq ce^{\gamma s}$, where c depand on A_p , p and γ*

This Lemma follows from Lemma 2.1 and remarks before Lemma 2.1.

REMARK 1. For any $-\infty < s < \infty$, let A_p^{is} be a bounded and $A_p^{is+it} = A_p^{is}A_p^{it}$.

If A_p^{is} is strongly continuous, then there exists constants $c > 0$ and $\gamma > 0$ such that

$$\| A^{is} \| \leq ce^{\gamma|s|}$$

in view of properties of groups of bounded operators.

REMARK 2. In the case where A_p is not invertible, R.Seely proved that

$$\begin{aligned} A_p^{-1}A_p f &= f - P_0 f, \qquad f \in \mathcal{D}(A_p) \\ \lim_{z\to 0} A_p^z f &= f - P_0 f, \qquad f \in L_p,\ Re\, z < 0 \end{aligned}$$

where $P_0 = \frac{1}{2\pi i}\int_{|\lambda|=\epsilon}(\lambda - A_p)^{-1}d\lambda$ be the projection on the generalized null space of A_p,and also proved that

$$\begin{aligned} S^z &= A_p^z + P_0 \text{ is a semigroup} \quad \text{and} \\ S^z &\longrightarrow I \quad \text{stronly} \quad \text{as } z \longrightarrow 0 \end{aligned}$$

while Lemma 2.1 is noted in terms of

$$\| S^{is} \| \leq ce^{i|s|}.$$

The allowable Banach spaces in this note are the δ-convex space. We know that the space X is δ-convex if and only if the Hilbert transform is a bounded operator on $L^q(R, X)$, $1 < q < \infty$.

From the δ-convexity of $L^p(\Omega)$ and Lemma 2.2 we obtain the following theorem by using results of G.Dore and A.Venni.

THEOREM 2.1. *Let A_p be an operator defined by (2.3), then for any $f \in L^r(0, T : L^p(\Omega))$ the Cauch problem*

$$\begin{cases} u'(t) + A_p(t) = f(t) \\ u(0) = 0 \end{cases}$$

has a unique solution

$$u \in W^{1,r}(0, T : L^p(\Omega)) \cap L^r(0, T : W^{2,p}(\Omega) \cap W^{1,p}(\Omega)), \qquad 1 < p < \infty$$

and

$$u \in W^{1,r}(0, T : L^1(\Omega)) \cap L^r(0, T : W^{2,1}(\Omega) \cap W^{1,q}(\Omega)), \qquad 1 \leq q \leq \frac{n}{n-1},\ p = 1.$$

REMARK 3. Let $[\cdot,\cdot]_{\frac{1}{2}}$ be a complex interpolace space, then $[W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega), L^p(\Omega)]_{\frac{1}{2}} = W_0^{1,p}$.
As is seen in [11,Lemma 5.5.1],we have

$$\begin{aligned} W^{1,2}(0,T:L^p(\Omega))\cap L^2(0,T:W^{2,p}(\Omega)\cap W^{1,p}(\Omega)) \\ \subset C([0,T]:W^{1,p}) \\ \subset C([0,T]:(\mathcal{D}(A_p)\cap L^p(\Omega))_{\frac{1}{2},2}) \end{aligned}$$

where $\mathcal{D}(A^p) = W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$

3. The relation between stabilizability of solution of (3.1) and controllability

Let U be complex Banach spaces.
We consider the following equation

$$\text{(3.1)} \qquad \begin{cases} \dfrac{d}{dt}u(t) = A_p u(t) + \Phi f(t) \\ x(0) = u_0 \end{cases}$$

where A_p is the operator in section 2 and $\Phi \in B(u, W_0^{1,p}(\Omega))$.
We assume

$$\text{(3.2)} \qquad \sigma(A_p) \cap \{\lambda : Re\lambda = 0\} = \phi.$$

Set $\sigma_+ = \sigma(A_p)\cap\{\lambda : Re\lambda > 0\}$, $\sigma_- = \sigma(A_p)\cap\{\lambda : Re\lambda < 0\}$
We assume also that

$$\text{(3.3)} \qquad \sigma_+ = \{\lambda_1,\cdots,\lambda_N\}$$

$$\text{(3.4)} \qquad -w_0 = \sup\{Re\lambda : \lambda\in\sigma_-\} < 0$$

and for each $j = 1,\cdots,N$, the spectral projection

$$\text{(3.5)} \qquad P_{\lambda_i} = \frac{1}{2\pi i}\int_{\Gamma_{\lambda_j}} (\lambda - A_p)^{-1}\, d\lambda$$

is the projection on generalized null space of A_p with finite rank, where Γ_{λ_j} is a small circle centered at λ_j such that it surrounds no point of $\sigma(A_p)$ except λ_j.

If A_p has a compact resolvent, then above assumptions is hold.

If $\lambda_j \in \sigma_+ \quad j = 1, \cdots, N$, then we have the Laurent expansion at $\lambda = \lambda_j$ is

$$(\lambda - A)^{-1} = \sum_{n=0}^{K_{\lambda_j}-1} \frac{Q^n_{\lambda_j}}{(\lambda - \lambda_j)^{n+1}} + R_0(\lambda)$$

where $Q^0_{\lambda_j} = P_{\lambda_j}$, $Q_{\lambda_j} = (A - \lambda_j)P_{\lambda_j}$ and $R_0(\lambda)$ is the analytic part of $(\lambda - A)^{-1}$ at $\lambda = \lambda_j$, hence λ_j is a pole of $(\lambda - A)^{-1}$ whose order is denoted by K_{λ_j}.

Moreover the order of a pole $\bar{\lambda}$ of $(z - A_p^*)^{-1}$ is equal to K_λ.

The above operator Q_{λ_j} is defind by

$$Q_{\lambda_j} = \frac{1}{2\pi i} \int_{\Gamma_\lambda} (u - \lambda)(\lambda - A)^{-1} du$$

We have $Q_\lambda^{K_{\lambda_j}} = 0, \quad Im\, Q_{\lambda_j} \subset Im\, P_{\lambda_j}$.

We put $X_{\lambda_j} = Im\, P_{\lambda_j}$.

Let $\Phi f \in L^2(0, \infty : W_0^{1,p}(\Omega))$ and $u(t)$ be the mild solution of the equation (3.1) i.e.,

$$u(t) = s(t)u_0 + \int_0^t S(t-s)\Phi f(s) ds.$$

We define the sets of attainability R and the unobservable N by

$$R = \{ \int_0^t S(t-s)\Phi u(s) : t \geq 0,\, u \in L^2(0, \infty : U)\} \subset W_0^{1,p}(\Omega).$$

$$N = \bigcap_{t \geq 0} Ker \Phi^* S^*(t) \subset L^{p'}(\Omega)$$

DEFINITION 3.1.
(1) For any $\lambda \in \sigma_+$, $S(t) = e^{tA_p}$ *is* $\lambda - controllable$ *if* $Cl(R) \supset X_\lambda$
(2) For any $\lambda \in \sigma_+$, $S^*(t) = e^{tA'_p}$ *is* $\lambda - observable$ *if* $N \cap X^*_{\bar\lambda} = \{0\}$

The following Lemma is proved in [7. Theorem 7.2].

LEMMA 3.1. *For any* $\lambda \in \sigma_+$

$$\begin{aligned} L^p(\Omega) &= Ker(\lambda - A_p)^{K_\lambda} \oplus Im(\lambda - A_p)^{K_\lambda}, \\ X_\lambda &= ImP_\lambda = Ker(\lambda - A_p)^{K_\lambda} \\ L^{p'}(\Omega) &= Ker(\bar\lambda - A_{p'})^{K_\lambda} \oplus Im(\bar\lambda - A_{p'})^{K_\lambda}, \\ X^*_{\bar\lambda} &= ImP^*_{\bar\lambda} = Ker(\bar\lambda - A_{p'})^{K_\lambda}. \end{aligned}$$

If λ is a pole of $(z - A)^{-1}$ then above Lemma is also hold.

LEMMA 3.2. *The following statements are equivalent ;*
For any $\lambda \in \sigma_+$

$$(i) S(t)\ is\ \lambda - controllable$$
$$(ii) S^*(t)\ is\ \bar\lambda - observable$$

Proof. By the Hahn-Banach theorem,the necessary and sufficuently condition for (1) is that $R^\perp \subset X^\perp_\lambda$.
From the Lemma 3.1 and duality theorem

$$\begin{aligned} X^\perp_\lambda &= (ImP_\lambda)^\perp = KerP^*_{\bar\lambda} = Im(\bar\lambda - A_{p'})^{K_\lambda} \\ R^\perp &= (\bigcup_{t\geq 0}\{\int_0^t S(t-s)\Phi u(s)ds : u \in L^2(0,\infty : U)\})^\perp \\ &= \bigcap_{t\geq 0}\{\int_0^t S(t-s)\Phi u(s)ds : u \in L^2(0,\infty : U)\}^\perp \\ &= \bigcap_{t\geq 0} Ker\Phi^\perp S^\perp(t) \\ &= N \end{aligned}$$

Hence in view of Lemma 3.1.

$$N \cap X^*_{\bar\lambda} = Im(\bar\lambda - A_{p'})^{K_\lambda} \cap X^*_{\bar\lambda} = \{0\}$$

LEMMA 3.3. *The following statements are equivalent ;*

$$(i) S^*(t) \text{ is } \bar\lambda - observable$$

$$(ii)\left(\bigcap_{j=1}^{K_\lambda - 1} Ker\,\Phi^*(Q^*_{\bar\lambda})\right) \cap X^*_{\bar\lambda} = \{0\}$$

Proof. For each $f \in X^*_{\bar\lambda}$, then $g = P^*_{\bar\lambda} g$ and

$$\begin{aligned} S^*(t)f = S^*(t)P^*_{\bar\lambda}g &= \frac{1}{2\pi i}\int_{\Gamma_{\bar\lambda}} e^{zt}(z - A_{p'})^{-1} g\,dz \\ &= e^{\bar\lambda t}\frac{1}{2\pi i}\int_{\Gamma_{\bar\lambda}} e^{(z-\bar\lambda)t}(z - A_{p'})^{-1} g\,dz \\ &= e^{\bar\lambda t}\left\{\sum_{n=0}^{K_\lambda - 1}\frac{t^n}{n!}\left(\frac{1}{2\pi i}\int_{\Gamma_{\bar\lambda}}(z-\bar\lambda)^n(z - A_{p'})^{-1} g\,dz\right)\right\} \\ &= e^{\bar\lambda t}\sum_{n=0}^{K_\lambda - 1}\frac{t^n}{n!}(Q^*_{\bar\lambda_j})^n g \end{aligned}$$

Hence, if $f \in N \cap X^*_{\bar\lambda}$, in view of above result we can see

$$e^{\bar\lambda t}\sum_{n=0}^{K_\lambda - 1}\frac{t^n}{n!}\Phi^*(Q^*_{\bar\lambda})^n g = 0, \quad t \geq 0.$$

Therefore $g \in (\bigcap_{j=0}^{K_\lambda - 1} Ker\Phi^{\perp}(Q^*_{\bar\lambda_j})^n) \cap (X^*_{\bar\lambda})$, here we used that $e^{\lambda t}\sum_{n=0}^{K_\lambda - 1}\frac{t^n}{n!}\Phi^*(Q^*_{\bar\lambda_j})^n g = 0,\ t \geq 0$ if and only if $\Phi^*(Q^*_{\bar\lambda_j})^n g = 0,\ n = 0, \cdots, K_\lambda - 1$.

Now we consider the stabilizability problem for (3.1). A necessary and sufficient consider for stabilizability is given by [5. proposition 3.1].

The following theorem is the relation between the properties of controllability and stabilizability for solution of (3.1).

THEOREM 3.1. *The following statements are equivalent;*

(i) For any $g \in L^p(\Omega)$, *there exists* $f \in L^2(0,\infty : U)$ *such that the mild solution of (3.1) belongs to* $L^2(0,\infty : L^p(\Omega))$

(ii) $S(t)$ *is* λ_j*-controllable* $j = 1,\cdots,N$

(iii) $S^*(t)$ *is* $\bar{\lambda}_j$*-observable* $j = 1,\cdots,N$.

Proof. It follows from [5. proposition 3.1] that the necessary and sufficient condition (i) is that for $j = 1,\cdots,N$.

$$\{x^* \in X^*_{\bar{\lambda}_j} : \Phi^*(A_{p'} - \bar{\lambda}_j)^K x^* = 0, \ \ K = 0,\cdots,m_j - 1\} = \{0\}.$$

If $x^* \in X^*_{\bar{\lambda}_j}$, then $P^*_{\bar{\lambda}_j} x^* = x^*$ and

$$\begin{aligned}\Phi^*(A_{p'} - \bar{\lambda}_j)^K x^* &= \Phi^*(A_{p'} - \bar{\lambda}_j)^K P^*_{\bar{\lambda}_j} x^* \\ &\le \Phi^*(Q^*_{\bar{\lambda}_j})^K x^*\end{aligned}$$

By virtue of Lemma 3.3 (i) is equivalent to (iii)

Hence this theorem follows from Lemma 3.2.

4. Application for retarded system

In this section we are interested in the retared functional differential equation

(4.1)

$$\begin{cases} \dfrac{du(t)}{dt} = A_0 u(t) + A_1 u(t-h) + \displaystyle\int_{-h}^{0} a(s) A_2 u(t+s) ds + \Phi_0 f(t) \\ u(0) = g^0, \ \ u(s) = g'(s) \quad a.e \ \ s \in [-h, 0), \ h > 0 \end{cases}$$

where $g = (g^0, g') \in W_0^{1,p}(\Omega) \times L^2(-h, 0 : W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))$,

$f \in L^2(0,\infty : U)$, $\Phi_0 \in B(c, W_0^{1,p}(\Omega))$, $A_i \in B(\mathcal{D}(A_p), L^p(\Omega))$ and $A_0 = A_p$.

In particular, if p=1, then $g \in W_0^{1,q}(\Omega) \times L^2(-h, 0 : W_0^{1,2}(\Omega) \cap W_0^{1,q}(\Omega))$ for $1 \le q \le n/n-1$.

Let $Z_p = W_0^{1,p}(\Omega) \times L^2(-h, 0 : \mathcal{D}(A_p))$ with norm

$$\| g \|_2 = (| g^0 |_{W_0^{1,p}(\Omega)} + \int_{-h}^{0} | g(s) |^2_{\mathcal{D}(A_p)} ds)^{\frac{1}{2}}, \ \ g \in Z_p.$$

Let $S_A(t); Z_p \longrightarrow M$ be the solution semigroup for (4.1) defined $S_A(t)g = (u(t; g), u(t-0) : g))$. for any $g \in Z$ where $u(t; g)$ is the mild solution of (4.1) (See [5]).

We can rewrite (4.1) as

$$(4.2) \qquad \begin{cases} \dfrac{d}{dt}x(t) = Ax(t) + \Phi f(t) \\ x(0) = g = (g^0, g') \end{cases}$$

in the space Z, where $S_A(t) = e^{tA}$ and Φ is the operator defined by $\Phi f = (\Phi_0 f, 0)$

In what follows we consider the operator A with the assumptions (3.2) - (3.4), but A_p need not be satisfied in this case.

In the remainder of this section by P_λ and Q_λ we denote the operator mentioned in section with A_p replaced by A.

It is easily seen that the whole contents of section 3 in vailed for the system (4.2) as in seen in [8] with this remark we have ;

THEOREM 4.1. *The following statments are equivalent;*

(i) For any $g \in Z$, there exists $f \in L^2(0, \infty : U)$ such that the mild solution u of (4.1) satisfies

$$\int_0^\infty \{| u(t) |^2_{W_0^{1,p}(\Omega)} + \int_{-h}^0 | u(t+s) |^2_{D(A_p)} ds\} dt < \infty$$

(ii) $Ker(\bar{\lambda}_j - A_{p'}) \cap Ker\Phi^* = \{0\}, \ j = 1, \cdots, N.$

Proof. From Theorem 3.1, it follows that (i) is equivalent to the fact that $S_A^*(t)$ is $\bar{\lambda}_j$ -observable, $j = 1, \cdots, N$

We have only to prove that the condition (ii) is equivalent that $S^*(t)$ is $\bar{\lambda}_j$-observable.

Let (ii) be hold and

$$\phi \in (\bigcap_{j=0}^{K_\lambda - 1} Ker\Phi^*(Q^*_{\bar{\lambda}_j})^j) \cap Z^*_{\bar{\lambda}_j}, \ Z^*_{\bar{\lambda}_j} = ImP^*_{\bar{\lambda}_j} Z$$

then $\phi \in Ker(\bar{\lambda} - A_{p'})^{K\lambda}$ and $\Phi^*(Q^*_{\bar{\lambda}_j})\phi = 0$ for $j = 0, \cdots, K_\lambda - 1$ here we used that $(Q^*_{\bar{\lambda}_j})^j = (A_{p'} - \bar{\lambda}_j)^j P^*_{\bar{\lambda}_j}$ for $0 \leq j \leq K_\lambda - 1$ and $P^*_{\bar{\lambda}}\phi - 0$.

We put $\phi_1 = (\bar{\lambda}_j - A_{p'})^{K_\lambda - 1}\phi$ then $\phi_1 \in Ker(\bar{\lambda}_j - A_{p'})$,

$$\Phi^*\phi_1 = \Phi^*(\bar{\lambda} - A_{p'})^{K_\lambda - 1}\phi = 0$$

In view of (ii) we have $\phi_1 = 0$.
Let $\phi_2 = (\bar{\lambda} - A_{p'})^{K_\lambda - 1}\phi$ then $\phi_2 \in Ker(\bar{\lambda} - A_{p'})$ and $\Phi^*\phi_2 = \Phi^*(\bar{\lambda} - A_{p'})^{K_\lambda - 2}\phi = 2$, hence we have $\phi_2 = 0$.
Continuing this procedure, we conclude that $\phi = 0$.
Conversely, let $\phi \in Ker(\bar{\lambda}_j - A_{p'}) \cap Ker\Phi^*$ and $(\bigcap_{j=0}^{K_\lambda - 1} Ker\Phi^*(Q^*_{\bar{\lambda}_j})^j) \cap Z^*_{\bar{\lambda}_j}\{0\}$ then $\phi \in Z^*_{\bar{\lambda}_j}$ and $A_{p'}\phi = \bar{\lambda}_j\phi$.
Hence, since $\phi \in Ker\Phi^*$,

$$\Phi^*S^*(t)\phi = \Phi^*(e^{\bar{\lambda}t}\phi) = e^{\bar{\lambda}t}\Phi^*\phi = 0, \ t \geq 0.$$

therefore $\phi \in Ker\Phi^*S^*(t) \cap Z^*_{\bar{\lambda}_j} = (\bigcap_{j=0}^{K_\lambda - 1} Ker\Phi^*(Q^*_{\bar{\lambda}_j})^j) \cap Z^*_{\bar{\lambda}_j} = \{0\}$.

References

1. S. Agmon, *On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems*, Comm. Pure. Appl. Math. **15** (1963), 119–147.
2. D. L. Burkholder, *A geometrical charaterization of Banach spaces in which martingale difference sequences are unconditional*, Ann. Probab. **9** (1981), 997–1011.
3. G. Da Prato and A. Lunardi, *Stabilizability of integrodifferential parabolic equations*, to appear J. Integ. Eqs. and Appl .
4. G. Dore and A. Venni, *On the closedness of the sum of two closed operators*, Math. Z. **196** (1987), 189-201.
5. J. M. Jeong, *Stabilizability of retared functional differential equation in Hilbert space*, to appear Osaka J. Math..
6. T. Kato, *Pertubation theory 2nd Ed.*, Springer, Berlin-Heidelberg-New York, 1976.
7. S. Nakagiri, *Structural properties of functional differential equations in Banach spaces*, Osaka J Math **25** (1988), 353–398.
8. S. Nakagiri and M. Yamamoto, *Controllability and observability of linear retared systems in Banach spaces*, to appear..
9. R. Seely, *Norms and domains of the complex power A_B^Z*, Amer. J. Math. **93** (1971), 299-309.
10. T. Suzuki and M. Yamamoto, *Observability, Controllability and feedback stabilizability for evolution equations I*, Japan J. Appl .Math. **2** (1985), 211–228.
11. T. Tanabe, *Equations of evolution*, Pitman, London, 1979.
12. R. Triggiani, *Extentions of rank conditions for controllability and observability to Banach spaces and unbounded operators*, SIAM. J. Control Optim **14** (1976), 313–338.

13 K. Yosida, *Functional analysis 3rd Ed.*, Springer, Berlin-Gottingen-Heidelberg, 1980.

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