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EXTENSIONS OF SOME COMMON FIXED POINT THEOREMS

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1. Introduction

In 1976, G. Jungck [4] initially proved a common fixed point theorem for commuting mappings, which generalizes the well-known Banach's fixed point theorem. This result has been generalized, extended and improved in various ways by many authors ([3], [5]–[7] and [9]–[13]).

Recently, S. Sessa [11] introduced a generalization of commutativity, which is called weak commutativity, and prove some common fixed point theorems for weakly commuting mappings, which generalize the result of K. M. Das and K. V. Naik [3]. On the other hand, G. Jungck [5] introduced the concept of more generalized commutativity, so called compatibility, which is more general than that of weak commutativity. The utility of compatibility in the context of fixed point theory was initially demonstrated in extending a theorem of S. Park and J. S. Bae [10]. In general, commuting mappings are weakly commuting and weakly commuting mappings are compatible, but the converses are not necessarily true as in the examples of the section 2. By employing compatible mappings in stead of commuting mappings and using four mappings as opposed to three, G. Jungck [6] extended the results of M. S. Khan and M. Imdad [9], S. L. Singh and S. P. Singh [13] and, quite recently, also obtained an interesting result concerning with his concept in his consecutive paper [7].

In this paper, we prove some common fixed point theorems for compatible mappings, which extend the results of D. E. Anderson, K. L. Singh and J. H. M. Whitfield [1] and G. Jungck [6], and give some examples to illustrate our main theorems.

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2. Preliminaries

In this section, we introduce some definitions and some examples related to the definitions. Also, we give some lemma for our main theorems.

For some definitions, terminologies and notations in this paper, we refer to [1], [5], [11] and [12].

DEFINITION 2.1. Let A and B be mappings from a metric space (X, d) into itself. Then the pair (A, B) is said to be ***asymptotically regular*** at x_0 in X if $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, where $\{x_n\}$ is a sequence in X defined by $x_1 = Ax_0$, $x_2 = Bx_1, \cdots$, $x_{2n+1} = Ax_{2n}$, $x_{2n+2} = Bx_{2n+1}, \cdots$.

DEFINITION 2.2. Let A and B be mappings from a metric space (X, d) into itself. Then A and B are said to be *weakly commuting mappings* on X if $d(ABx, BAx) \leq d(Ax, Bx)$ for all x in X.

Clearly, any commuting mappings are weakly commuting, but the converse is not necessarily true as in the following example:

EXAMPLE 2.3. Let $X = [0, 1]$ with the Euclidean metric d. Define A and $B : X \to X$ by

$$Ax = \frac{1}{2}x \quad \text{and} \quad Bx = \frac{x}{2+x}$$

for all x in X. Then we have, for any x in X,

$$\begin{aligned} d(ABx, BAx) &= \frac{x}{4+x} - \frac{x}{4+2x} \\ &\leq \frac{x^2}{4+2x} \\ &= d(Ax, Bx). \end{aligned}$$

Thus, A and B are weakly commuting mappings on X, but they are not commuting on X since

$$BAx = \frac{x}{4+x} > \frac{x}{4+2x} = ABx$$

for any non-zero x in X.

DEFINITION 2.4. Let A and B be mappings from a metric space (X,d) into itself. Then A and B are said to be ***compatible mappings*** on X if $\lim_{n\to\infty} d(ABx_n, BAx_n) = 0$ when $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$ for some t in X.

Obviously, weakly commuting mappings are compatible, but the converse is not necessarily true as in the following example:

EXAMPLE 2.5. Let $X = (-\infty, \infty)$ with the Euclidean metric d. Define A and $B : X \to X$ by

$$Ax = x^3 \quad \text{and} \quad Bx = 2 - x$$

for all x in X. Since $d(Ax_n, Bx_n) = |x_n - 1|\left|x_n^2 + x_n + 2\right| \to 0$ iff $x_n \to 1$,

$$\lim_{n\to\infty} d(BAx_n, ABx_n) = \lim_{n\to\infty} 6\,|x_n - 1|^2 = 0$$

as $x_n \to 1$. Thus, A and B are compatible on X, but they are not weakly commuting mappings on X since

$$d(ABx, BAx) = 6 > 2 = d(Ax, Bx)$$

for $x\,(= 0)$ in X.

Thus, commuting mappings are also compatible, but the converse is not necessarily true.

We need the following lemmas for our main theorems :

LEMMA 2.6. *Let A and B be compatible mappings from a metric space (X,d) into itself. Suppose that $At = Bt$ for some t in X. Then $d(ABt, BAt) = 0$.*

LEMMA 2.7. *Let A and B be compatible mappings from a metric space (X,d) into itself. Suppose that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$ for some t in X. Then $\lim_{n\to\infty} BAx_n = At$ if A is continuous.*

3. Fixed point theorems

In this section, we give main theorems and so some corollaries proved by some authors are obtained.

Drawing inspiration from the contractive condition of L. B. Ćirić [2], let A, B, S and T be mappings from a metric space (X, d) into itself satisfying the following conditions:

$$A(X) \subset T(X) \quad \text{and} \quad B(X) \subset S(X), \tag{3.1}$$

$$\begin{aligned} d(Ax, By) \leq h \max\{&d(Ax, Sx), d(By, Ty), \\ &d(Ax, Ty), d(By, Sx), d(Sx, Ty)\} \end{aligned} \tag{3.2}$$

for all x, y in X, where $0 \leq h < 1$. Then for an arbitrary point x_0 in X, by (3.1), we can choose a point x_1 in X such that $Tx_1 = Ax_0$ and for this point x_1, there exists a point x_2 in X such that $Sx_2 = Bx_1$ and so on. Continuing in this manner, we can define a sequence $\{y_n\}$ in X such that

$$y_{2n+1} = Tx_{2n+1} = Ax_{2n} \quad \text{and} \quad y_{2n} = Sx_{2n} = Bx_{2n-1} \tag{3.3}$$

for $n = 0, 1, 2, \cdots$.

DEFINITION 3.1. Let A, B, S and T be mappings from a metric space (X, d) into itself. Then the pair (A, B) is said to be *asymptotically regular* at x_0 in X with respect to S and T if $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$, where $\{y_n\}$ is the sequence in X defined by (3.3).

If S and T are the identity mappings on X, then Definition 3.1 becomes Definition 2.1.

LEMMA 3.2. *Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying the conditions* (3.1) *and* (3.2). *Suppose that*

(3.4) *the pair (A, B) is asymptotically regular at some point in X with respect to S and T.*

Then the sequence $\{y_n\}$ defined by (3.3) *is a Cauchy sequence in X.*

Proof. Let the pair (A, B) be asymptotically regular at x_0 in X with respect to S and T. Then we have

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0, \tag{3.5}$$

where $\{y_n\}$ is the sequence in X defined (3.3).

In order to show that $\{y_n\}$ is a Cauchy sequence in X, it is sufficient to show that $\{y_{2n}\}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}$ is not a Cauchy sequence. Then there is an $\epsilon > 0$ such that for each even integer $2k$, there exist even integers $2m(k)$ and $2n(k)$ with $2m(k) > 2n(k) \geq 2k$ such that

$$d(y_{2m(k)}, y_{2n(k)}) > \epsilon. \tag{3.6}$$

For each even integer $2k$, let $2m(k)$ be the least even integer exceeding $2n(k)$ satisfying (3.6), that is,

$$d(y_{2n(k)}, y_{2m(k)-2}) \leq \epsilon \quad \text{and} \quad d(y_{2n(k)}, y_{2m(k)}) > \epsilon. \tag{3.7}$$

Then for each even integer $2k$,

$$\begin{aligned}\epsilon &< d(y_{2n(k)}, y_{2m(k)}) \\ &\leq d(y_{2n(k)}, y_{2m(k)-2}) + d(y_{2m(k)-2}, y_{2m(k)-1}) + d(y_{2m(k)-1}, y_{2m(k)}).\end{aligned}$$

Therefore, by (3.5) and (3.7), we have

$$\lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)}) = \epsilon. \tag{3.8}$$

From the triangle inequality, it follows that

$$\left|d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})\right| \leq d(y_{2m(k)-1}, y_{2m(k)})$$

and

$$\begin{aligned}&\left|d(y_{2n(k)+1}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})\right| \\ &\qquad\qquad \leq d(y_{2m(k)-1}, y_{2m(k)}) + d(y_{2n(k)}, y_{2n(k)+1}).\end{aligned}$$

Hence, by (3.5) and (3.8), as $k \to \infty$,

$$d(y_{2n(k)}, y_{2m(k)-1}) \;\to\; \epsilon \quad \text{and} \quad d(y_{2n(k)+1}, y_{2m(k)-1}) \;\to\; \epsilon.$$

Now, by (3.2) and (3.3), we obtain

$$\begin{aligned}d(y_{2n(k)}, y_{2m(k)}) &\leq d(y_{2n(k)}, y_{2n(k)+1}) + d(Ax_{2n(k)}, Bx_{2m(k)-1}) \\ &\leq d(y_{2n(k)}, y_{2n(k)+1}) + h\max\{d(y_{2n(k)}, y_{2n(k)+1}), \\ &\qquad d(y_{2m(k)-1}, y_{2m(k)}), d(y_{2n(k)+1}, y_{2m(k)-1}), \\ &\qquad d(y_{2m(k)}, y_{2n(k)}), d(y_{2n(k)}, y_{2m(k)-1})\}.\end{aligned}$$

It follows from this fact that

$$\epsilon \leq h\max\{0, 0, \epsilon, \epsilon, \epsilon\} < \epsilon \quad \text{as } k \to \infty,$$

which is a contraction. Therefore, $\{y_n\}$ is a Cauchy sequence in X.

Now, we are ready to give our main theorems :

THEOREM 3.3. *Let A, B, S and T be mappings from a complete metric space (X,d) into itself satisfying the conditions* (3.1), (3.2) *and* (3.4). *Suppose that*

(3.9) *one of A, B, S and T is continuous, and*

(3.10) *the pairs A,S and B,T are compatible on X.*

Then A, B, S and T have a unique common fixed point in X.

Proof. Let $\{y_n\}$ be the sequence in X defined by (3.3). By Lemma 3.2, $\{y_n\}$ is a Cauchy sequence and hence it converges to some point z in X. Consequently, the subsequences $\{Ax_{2n}\}$, $\{Sx_{2n}\}$, $\{Bx_{2n-1}\}$ and $\{Tx_{2n-1}\}$ of $\{y_n\}$ also converge to z.

Now, suppose that S is continuous. Since A and S are compatible on X, Lemma 2.7 gives that

$$S^2x_{2n} \quad \text{and} \quad ASx_{2n} \to Sz \quad \text{as } n \to \infty.$$

By (3.2), we obtain

$$\begin{aligned} d(ASx_{2n}, Bx_{2n-1}) \leq h \max\{&d(ASx_{2n}, S^2x_{2n}), d(Bx_{2n-1}, Tx_{2n-1}), \\ &d(ASx_{2n}, Tx_{2n-1}), d(Bx_{2n-1}, S^2x_{2n}), \\ &d(S^2x_{2n}, Tx_{2n-1})\}. \end{aligned}$$

By letting $n \to \infty$, we have

$$d(Sz, z) \leq h \max\{0, 0, d(Sz,z), d(z,Sz), d(Sz,z)\},$$

so that $z = Sz$. By (3.2), we also obtain

$$\begin{aligned} d(Az, Bx_{2n-1}) \leq h \max\{&d(Az, Sz), d(Bx_{2n-1}, Tx_{2n-1}), \\ &d(Az, Tx_{2n-1}), d(Bx_{2n-1}, Sz), \\ &d(Sz, Tx_{2n-1})\}. \end{aligned}$$

By letting $n \to \infty$, we have

$$d(Az, z) \leq h \max\{d(Az,Sz), 0, d(Az,z), d(z,Sz), d(Sz,z)\},$$

so that $z = Az$. Since $A(X) \subset T(X)$, $z \in T(X)$ and hence there exists a point u in X such that $z = Az = Tu$.

$$\begin{aligned} d(z, Bu) &= d(Az, Bu) \\ &\leq h \max\{0, d(Bu,Tu), d(Az,Tu), d(Bu,z), d(Sz,Tu)\}, \end{aligned}$$

which implies that $z = Bu$. Since B and T are compatible on X and $Tu = Bu = z$, $d(TBu, BTu) = 0$ by Lemma 2.6 and hence $Tz = TBu = BTu = Bz$. Moreover, by (3.2), we obtain

$$\begin{aligned} d(z, Tz) &= d(Az, Bz) \\ &\leq h \max\{0, d(Bz, Tz), d(z, Tz), d(Bz, z), d(z, Tz)\}, \end{aligned}$$

so that $z = Tz$. Therefore, z is a common fixed point of A, B, S and T. Similarly, we can also complete the proof when T is continuous.

Next, suppose that A is continuous. Since A and S are compatible on X, it follows from Lemma 2.7 that

$$A^2 x_{2n} \quad \text{and} \quad SAx_{2n} \to Az \quad \text{as } n \to \infty.$$

By (3.2), we have

$$\begin{aligned} d(A^2 x_{2n}, Bx_{2n-1}) \leq h \max\{&d(A^2 x_{2n}, SAx_{2n}), d(Bx_{2n-1}, Tx_{2n-1}), \\ &d(A^2 x_{2n}, Tx_{2n-1}), d(Bx_{2n-1}, SAx_{2n}), \\ &d(SAx_{2n}, Tx_{2n-1})\}. \end{aligned}$$

By letting $n \to \infty$, we obtain

$$d(Az, z) \leq h \max\{0, 0, d(Az, z), d(z, Az), d(Az, z)\},$$

so that $z = Az$. Hence, there exists a point v in X such that $z = Az = Tv$.

$$\begin{aligned} d(A^2 x_{2n}, Bv) \leq h \max\{&d(A^2 x_{2n}, SAx_{2n}), d(Bv, Tv), \\ &d(A^2 x_{2n}, Tv), d(Bv, SAx_{2n}), \\ &d(SAx_{2n}, Tv)\}. \end{aligned}$$

By letting $n \to \infty$, we have

$$d(z, Bv) \leq h \max\{0, d(Bv, Tv), d(Az, Tv), d(Bv, z), d(z, Tv)\},$$

which implies that $z = Bv$. Since B and T are compatible on X and $Tv = Bv = z$, $d(TBv, BTv) = 0$ by Lemma 2.6 and hence $Tz = TBv = BTv = Bz$. Moreover, by (3.2), we have

$$\begin{aligned} d(Ax_{2n}, Bz) \leq h \max\{&d(Ax_{2n}, Sx_{2n}), d(Bz, Tz), \\ &d(Ax_{2n}, Tz), d(Bz, Sx_{2n}), d(Sx_{2n}, Tz)\}. \end{aligned}$$

By letting $n \to \infty$, we obtain

$$d(z, Bz) \leq h \max\{0, d(Bz, Tz), d(z, Tz), d(Bz, z), d(z, Tz)\},$$

so that $z = Bz$. Since $B(X) \subset S(X)$, there exists a point w in X such that $z = Bz = Sw$.

$$\begin{aligned} d(Aw, z) &= d(Aw, Bz) \\ &\leq h \max\{d(Aw, Sw), 0, d(Aw, z), d(z, Sw), d(Sw, z)\}, \end{aligned}$$

so that $Aw = z$. Since A and S are compatible on X and $Aw = Sw = z$, $d(SAw, ASw) = 0$ and hence $Sz = SAw = ASw = Az$. Therefore, z is a common fixed point of A, B, S and T. Similarly, we can also complete the proof when B is continuous.

Finally, in order to prove the uniqueness of z, suppose that z and w, $z \neq w$, are common fixed points of A, B, S and T. Then, by (3.2), we obtain

$$\begin{aligned} d(z, w) &= d(Az, Bw) \\ &\leq h \max\{0, 0, d(z, w), d(w, z), d(z, w)\} \\ &= h\, d(z, w), \end{aligned}$$

which is a contraction. Hence $z = w$. This completes the proof.

The following corollary follows immediately from Theorem 3.3 :

COROLLARY 3.4. ([1]) *Let A and B be mappings from a complete metric space (X, d) into itself satisfying the following condition :*

$$\begin{aligned} d(Ax, By) \leq h \max\{&d(Ax, x), d(By, y), \\ &d(Ax, y), d(By, x), d(x, y)\} \end{aligned}$$

for all x, y in X, where $0 \leq h < 1$. Suppose that the pair (A, B) is a asymptotically regular at some point in X.
Then A and B have a unique common fixed point in X.

THEOREM 3.5. *Let A, B, S and T be mappings from a complete metric space (X,d) into itself satisfying the conditions* (3.1), (3.9), (3.10) *and* (3.11) *:*

$$(3.11)\quad d(Ax,By) \le h\max\Big\{d(Ax,Sx), d(By,Ty), \frac{1}{2}\left[d(Ax,Ty)+d(By,Sx)\right], d(Sx,Ty)\Big\}$$

for all x, y in X, where $0 \le h < 1$.
Then A, B, S and T have a unique common fixed point in X.

Proof. In virtue of Lemma 3.1 in [6], if we replace (3.2) by (3.11), then the condition of asymptotic regularity can be dropped. As in the proof of Theorem 3.3, this conclusion follows easily.

REMARK 3.1. Theorems in this paper extend, generalize and improve a number of fixed point theorems for commuting mappings ([3], [4], [9] and [13]).

REMARK 3.2. Theorem 3.5 generalizes the result of G. Jungck [6] by using any one continuous mapping as opposed to the continuity of both S and T.

REMARK 3.3. Quite recently, in [8], G. Jungck, P. P. Murthy and Y. J. Cho introduced the concept of compatible mappings of type (A) in metric spaces and obtained some relations between compatible mappings and compatible mappings of type (A). Also, they proved that two concepts of compatibility and compatibility of type (A) are equivalent under some conditions.

If the given mappings in all the theorems of this section are continuous, all the theorems in this paper are still true even though the condition of the compatibility is replaced by the compatibility of type (A).

4. Examples

In this section, we give some examples to illustrate our main theorems.

In the following example, we show the existence of a common fixed point of four mappings with one continuous mapping as opposed to the continuity of both S and T:

EXAMPLE 4.1. Let $X = [0,1]$ with the Euclidean metric d. Define A, B, S and $T : X \to X$ by

$$Ax = 0, \ Bx = \begin{cases} \frac{1}{4} & \text{if } x = \frac{1}{2}, \\ \frac{1}{4}x & \text{if } x \neq \frac{1}{2}, \end{cases} Sx = x \text{ and } Tx = \begin{cases} 1 & \text{if } x = \frac{1}{2}, \\ x & \text{if } x \neq \frac{1}{2}, \end{cases}$$

for all x in X. Now, $A(X) \subset S(X)$ and $B(X) \subset T(X)$. It is easily seen that pairs A, S and B, T are commuting on X and hence they are compatible on X. Further, we have

$$\begin{aligned} d(Ax, By) &= \begin{cases} \frac{1}{4} = \frac{1}{3}\, d(By, Ty) & \text{if } y = \frac{1}{2}, \\ \frac{1}{4}x = \frac{1}{3}\, d(By, Ty) & \text{if } y \neq \frac{1}{2}, \end{cases} \\ &\leq \frac{1}{3} \max\{d(Ax, Sx), d(By, Ty), \\ &\qquad \frac{1}{2}[\, d(Ax, Ty) + d(By, Sx)\,], d(Sx, Ty)\} \end{aligned}$$

for all x, y in X. Thus, we see that all the hypotheses of Theorem 3.5 is satisfied. Here, zero is a common fixed point of A, B, S and T.

In the following example, we show the existence of a common fixed point of mappings which are compatible but not commuting :

EXAMPLE 4.2. Let $X = [1, \infty)$ with the Euclidean metric d. Define A, B, S and $T : X \to X$ by

$$Ax = x^3, \quad Bx = x^2, \quad Sx = 2x^6 - 1 \quad \text{and} \quad Tx = 2x^4 - 1$$

for all x in X. Now, $A(X) = B(X) = S(X) = T(X) = X$. Moreover, since $d(Ax_n, Sx_n) = |2x_n^3 + 1|\,|x_n^3 - 1| \ \to\ 0$ iff $x_n \to 1$,

$$\lim_{n\to\infty} d(ASx_n, SAx_n) = \lim_{n\to\infty} 6x_n^6(x_n^6 - 1)^2 = 0 \quad \text{as } x_n \to 1.$$

Thus, A and S are compatible on X, but they are not commuting mappings at $x = 2$. Likewise, since $d(Bx_n, Tx_n) = (2x_n^2 + 1)\,|x_n^2 - 1| \ \to\ 0$ iff $x_n \to 1$,

$$\lim_{n\to\infty} d(BTx_n, TBx_n) = \lim_{n\to\infty} 2(x_n^4 - 1)^2 = 0 \quad \text{as } x_n \to 1.$$

Further, we obtain

$$\begin{aligned} d(Ax, By) &\le \frac{1}{4}\, d(Sx, Ty) \\ &\le \frac{1}{4} \max\big\{d(Ax, Sx), d(By, Ty), \\ &\qquad\qquad \frac{1}{2}\,[\, d(Ax, Ty) + d(By, Sx)\,], d(Sx, Ty)\big\} \end{aligned}$$

since $d(Sx, Ty) = 2\,|x^3 - y^2|\,|x^3 + y^2| \ge 4\, d(Ax, By)$ for all x, y in X. Now, all the hypotheses of Theorem 3.5 except the commutativity of A and S is satisfied, but A, B, S and T have a unique common fixed point in X.

We give an example showing that Theorem 3.5 is no longer true if we do not assume that any one of mappings is continuous :

EXAMPLE 4.3. Let $X = [\,0, 1\,]$ with the Euclidean metric d. Define $A = B$ and $S = T : X \to X$ by

$$Ax = \begin{cases} \frac{1}{8} & \text{if } x = 0, \\ \frac{1}{8}x & \text{if } x \ne 0, \end{cases} \quad \text{and} \quad Sx = \begin{cases} 1 & \text{if } x = 0, \\ \frac{1}{2}x & \text{if } x \ne 0 \end{cases}$$

for all x in X. $A(X) = (0, \frac{1}{8}\,] \subset (0, \frac{1}{2}\,] \subset S(X)$. Moreover, we obtain

$$d(AS0, SA0) = \frac{1}{16} < \frac{7}{8} = d(S0, A0)$$

and $ASx = SAx = \frac{1}{16}\, x$ for all x in $X - \{0\}$. So, A and S are compatible on X. Further, we obtain

$$\begin{aligned} d(Ax, Ay) &= \begin{cases} 0 & \text{if } x = y = 0, \\ \frac{1}{8}(1 - x) < \frac{1}{4}(1 - \frac{1}{2}x) = \frac{1}{4}\, d(Sy, Sx) & \text{if } x > y = 0, \\ \frac{1}{8}(1 - y) < \frac{1}{4}(1 - \frac{1}{2}y) = \frac{1}{4}\, d(Sx, Sy) & \text{if } y > x = 0, \\ \frac{1}{8}\,|x - y| = \frac{1}{4}\, d(Sx, Sy) & \text{if } x, y \ne 0, \end{cases} \\ &\le \frac{1}{4} \max\big\{d(Ax, Sx), d(Ay, Sy), \\ &\qquad\qquad \frac{1}{2}\,[\, d(Ax, Sy) + d(Ay, Sx)\,], d(Sx, Sy)\big\} \end{aligned}$$

for all x, y in X. We find that all the hypotheses of Theorem 3.5 is satisfied except the continuity of A and S, but none of the given mappings has a fixed point in X.

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