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ON THE MINIMAL LEFT AND RIGHT IDEALS GENERATED BY AN IDEMPOTENT OF CONTINUOUS RINGS

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We already knew the following : if a ring R is semiprime and e is an idempotent of R, then Re is minimal left ideal if and only if eR is minimal right ideal. In this paper, we will get similar result as above statement for continuous ring.

We state some definitions and facts which were already well-known.

An element a in a ring R is said to be *left quasi-regular* if there exists r in R such that $r + a + ra = 0$. The element r is called the *left quasi-inverse* of a. A (left, right or two-sided) ideal I of R is said to be *left quasi-regular* if every element of I is left quasi-regular. Similarly, an element a in R is said to be *right quasi-regular* if there exists r in R such that $r + a + ar = 0$. *Right quasi-inverse* and *right quasi-regular ideals* can also be defined analogously. It can be easily verified that an element a in R is left (resp. right) quasi-regular if and only if $1 + a$ is left (resp. right) invertible.

THEOREM 1. *For a ring R, there exists a two-sided ideal $J(R)$ of R satisfying following equivalent conditions :*

(1) *$J(R)$ is the intersection of all annihilators of simple left R-modules.*

(2) *$J(R)$ is the intersection of all maximal ideals of R.*

(3) *$J(R)$ is a left quasi-regular left ideal which contains every left quasi-regular left ideal of R.*

Statements (1), (2) and (3) in the above theorem can also be true if "left" is replaced by "right". The ideal $J(R)$ of R is called the *Jacobson radical* of R. If e is a non-zero idempotent in R, then e is not in the Jacobson radical $J(R)$ because $1 - e$ is not invertible.

For a typical example of the Jacobson radical, let $F[[x]]$ be the formal power series ring over a field F. Then every ideal of the ring $F[[x]]$ has

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the form $x^k F[[x]]$ for some $k = 0, 1, 2, \ldots$. Thus in the ring $F[[x]]$, it follows that $xF[[x]]$ is the only one maximal ideal and so the Jacobson radical of the ring $F[[x]]$ is $xF[[x]]$.

A ring R is called *semiprimitive* if $J(R) = 0$. For any ring R, the ring $R/J(R)$ has the zero Jacobson radical by the correspondence between maximal left ideals of R and those of $R/J(R)$. Hence the ring $R/J(R)$ is always semiprimitive.

For a given left R-module M, the sum of all simple R-submodules of M is called the *socle* of M and is denoted by $Soc(M)$. Thereby M is completely reducible if and only if $M = Soc(M)$. Obviously $Soc(M)$ is invariant under any R-endomorphism of M. By F. Sandomierski [8], it is well-known that $Soc(M)$ is the intersection of all essential R-submodules of M.

Now for a ring R, the *left socle* $Soc_l(R)$ of R is the socle of the left R-module R. So $Soc_l(R)$ is a two-sided ideal of R. The *right socle* $Soc_r(R)$ of R is the socle of the right R-module R. Indeed, the right socle of R is the sum of all minimal right ideals of R. Also it is a two-sided ideal of R. If a submodule N of a module M is essential in M, then we have that $Soc(N) = Soc(M)$. When R is a semiprime ring, i.e., a ring without non-zero nilpotent two-sided ideal, $Soc_l(R)$ and $Soc_r(R)$ are coincided. But in general the left socle is not always equal to the right socle. For a typical example, let

$$R = \begin{pmatrix} F & F[x] \\ 0 & F[x] \end{pmatrix},$$

where F is a field and $F[x]$ is the polynomial ring with x indeterminate. Then the left socle of R is

$$\begin{pmatrix} 0 & F[x] \\ 0 & 0 \end{pmatrix}.$$

However the right socle of R is 0.

For a left R-module M,

$$Ann_R(M) = \{a \in R | am = 0 \quad \text{for all} \quad m \in M\}$$

is called the *annihilator* of M in R. It is easily verified that $Ann_R(M)$ is a two-sided ideal of R. The *left annihilator* $l(S)$ of a subset S in a ring R is the set of all a in R such that $as = 0$ for any s in S, that is, $l(S) = \{a \in R | as = 0 \quad \text{for all} \quad s \in S\}$. Obviously, $l(S)$ is a left ideal of R. If S is a left ideal of R, then $l(S)$ is two-sided ideal of R. Similarly the *right annihilator* $r(S)$ of S in R can be defined, that is, $r(S) = \{a \in R | sa = 0 \quad \text{for all} \quad s \in S\}$.

DEFINITION 2. A ring R is said to be *left continuous* if (1) every left ideal is essential in a left R-direct summand and (2) every left idel isomorphic to a direct summand is generated by an idempotent of R. A *right continuous* ring also can be defined similarly. A left and right continuous ring is called *continuous* ring. For an example of continuous ring, every princial ideal domain is continuous.

We already knew that the structure of minimal one-sided ideals was initially observed by Brauer.

LEMMA 3. *In a ring R, every minimal one-sided ideal is either nilpotent or generated by an idempotent.*

Proof. See [7].

According to Lemma 3, every minimal one-sided ideal of a semiprime ring is generated by an idempotent. Furthermore we have more information in semiprime ring case.

LEMMA 4. *Assume that R is a semiprime ring. Then for an idempotent e of R, Re is a mininal left ideal if and only if eR is a minimal right ideal.*

Proof. See [7].

Now for the case of continuous rings, we have a similar result as in Lemma 4.

THEOREM 5. *Assume that R is continuous ring. Then for an idempotent e of R, Re is a minimal left ideal if and only if eR is a minimal right ideal.*

Proof. Assume that Re is a minimal left ideal with e an idempotent in R. Then since the Jacobson radical $J(R)$ of R is the largest left quasi-regular ideal of R by Theorem 1, the idempotent e is not in $J(R)$. Now observe that either $Re \cap J(R) = 0$ or $Re \cap J(R) = Re$ because Re is a minimal left ideal. If $Re \cap J(R) = Re$, then e is in

$J(R)$, which is a constradiction. Hence $Re \cap J(R) = 0$. Now consider the left ideal

$$\bar{R}e = (Re + J(R))/J(R)$$

in the ring $\bar{R} = R/J(R)$, where $\bar{e} = e + J(R)$. Then $(Re + J(R))/J(R) = Re/(Re \cap J(R))$. But since $0 = Re \cap J(R)$, $\bar{R}e$ is a minimal left ideal of $\bar{R}$. By Lemma 4, $e\bar{R}$ is a minimal right ideal of the ring $\bar{R}$ because $\bar{R}$ is semiprime. In this case, we also have $eR \cap J(R) = 0$. Indeed,

$$eR \cap J(R) \subseteq ReR \cap J(R) \subseteq Soc_l(R) \cap J(R).$$

Since $J(R)Soc_l(R) = 0$ by Theorem 1, we have

$$(ReR \cap J(R)^2 \subseteq (Soc_l(R) \cap J(R))^2 = 0.$$

For more informations of $Soc_l(R)$, we write by Lemma 3

$$Soc_l(R) = \sum_{i \in H} Re_i + N,$$

where $\{Re_i | i \in H\}$ is the set of all minimal left ideals generated by idempotent e_is and N is the sum of all minimal nilpotent ideals of R. Then of course, we have

$$\sum_{i \in H} Re_i \cap N = 0.$$

Moreover, $\sum_{i \in H} Re_i$ is a two-sided ideal of R. In fact, let a be in R and consider the left ideal $Re_i a$ for any $i \in H$. Then $Re_i a$ is a left ideal which is an R-module epimorphic image of Re_i. Hence either $Re_i a = 0$ or $Re_i a$ is R-module isomorphic to Re_i because Re_i is a simple left R-module. If $Re_i a = 0$, then of course $Re_i a$ is contained in $\sum_{i \in H} Re_i$.

On the other hand, if $Re_i a$ is isomorphic to Re_i, then $Re_i a$ is generated by an idempotent because R is continuous. Thus we have that $Re_i a$ is contained in $\sum_{i \in H} Re_i$, and hence $\sum_{i \in H} Re_i$ is a two-sided ideal of R. Now from the fact that $(Soc_l(R) \cap J(R))^2 = 0$, we have $ReR \cap J(R) \subseteq N$ and so $ReR \cap J(R) = ReR \cap N$. But since $\sum_{i \in H} Re_i$ is a two-sided ideal of R, $ReR \cap N \subseteq \sum_{i \in H} Re_i \cap N = 0$. Thus $ReR \cap J(R) = 0$ and hence $eR \cap J(R) = 0$. Since $e\bar{R} = (eR + J(R))/J(R) \cong eR/(eR \cap J(R))$ is a minimal right ideal of $\bar{R}$ and $eR \cap J(R) = 0$, it follows that eR is a minimal right ideal of R. Now by the same argument, we can also show that Re is a minimal left ideal of R whenever eR is a minimal right ideal of R.

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