Pusan Kyŏngnam Math. J. 8(1992), No. 1, pp. 69–73

ON COSYMPLECTIC MANIFOLDS

Byung Hak Kim

Dedicated to Professor Kyu Chang Nam on his sixtieth birthday

1. Introduction

As a complex analogue to the Weyl conformal curvature tensor, S.S. Eum [2] introducted the so-called cosymplectic Bochner curvature tensor and studied its fundamental properties.

Recently, the cosymplectic manifolds have studied by G.D. Ludden [5] in the theory of submanifolds and T. Kashiwada [3] studied some conditions in order that the Bochner curvature tensor vanishes in the Kaehler manifolds. Similar studies were made by M. Seino [6] in the contact Bochner curvature tensor case.

The purpose of this paper is to study necessary and sufficient conditions for the cosymplectic Bochner curvature tensor to vanish in the cosymplectic manifolds.

2. Cosymplectic Bochner curvature tensor

Let M be an m-dimensional cosymplectic manifold with structure (ϕ, ξ, η, g), that is, a manifold M which admits a 1-form η, a vector field ξ, a metric tensor g satisfying

$$\begin{aligned} &\eta(\xi) = 1, \quad \phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \\ &g(\xi, X) = \eta(X), \quad g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) \end{aligned} \tag{2.1}$$

$$\nabla_X \phi = 0, \quad \nabla_X \xi = 0 \tag{2.2}$$

for any vector fields X and Y, where ∇ denotes the Riemannian connection of g. The fundamental 2-form Φ is defined by

$$\Phi(X,Y) = g(\phi X, Y).$$

Received June 11, 1992.

The cosymplectic Bochner curvature $B(X,Y,Z,U) = g(B(X,Y)Z,U)$ is defined by

(2.3)

$$\begin{aligned}
B(X,Y,Z,U) &= K(X,Y,Z,U) - [S(Y,Z)\{g(X,U)-\eta(X)\eta(U)\}\\
&- S(X,Z)\{g(Y,U)-\eta(Y)\eta(U)\} + S(X,U)\{g(Y,Z)-\eta(Y)\eta(Z)\}\\
&- S(Y,U)\{(g(X,Z)-\eta(X)\eta(Z)\} - \Phi(X,U)H(Z,Y)\\
&+ \Phi(Y,U)H(Z,X) - \Phi(Y,Z)H(U,X) + \Phi(X,Z)H(U,Y)\\
&+ 2\Phi(Z,U)H(Y,X) + 2\Phi(X,Y)H(U,Z)]/(m+3)\\
&+ Q[\{g(X,U)-\eta(X)\eta(U)\}\cdot\{g(Y,Z)-\eta(Y)\eta(Z)\}\\
&- \{g(Y,U)-\eta(Y)\eta(U)\}\cdot\{g(X,Z)-\eta(X)\eta(Z)\}\\
&- \Phi(X,U)\Phi(Z,Y) + \Phi(Y,U)\Phi(Z,X)\\
&+ 2\Phi(X,Y)\Phi(U,Z)]/(m+1)(m+3),
\end{aligned}$$

where

$$H(X,Y) = S(\phi X,Y) = -H(Y,X),$$
$$K(X,Y,Z,U) = g(K(X,Y)Z,U)$$

is the Riemannian curvature tensor, S is the Ricci curvature and Q is the scalar curvature of M.

It is easily seen that the cosymplectic Bochner curvature tensor satisfies the following conditions;

$$\begin{aligned}
&B(X,Y,Z,U) = -B(Y,X,Z,U),\\
&B(X,Y,Z,U) = -B(X,Y,U,Z),\\
(2.4)\qquad &B(X,Y,Z,U) = B(Z,U,X,Y),\\
&B(X,Y,Z,U) + B(Y,Z,X,U) + B(Z,X,Y,U) = 0,\\
&B(X,Y,\phi Z,Z) = 0, \quad B(\xi,X,Y,Z) = 0.
\end{aligned}$$

Let $q = (m-1)/2$ and take a ϕ - basis $\{e_1,\cdots,e_m\}$ at each point of M such that $e_1,\cdots,e_q,e_{q+1} = \phi e_1,\cdots,e_{2q} = \phi e_q, e_m = \xi$. We write the basis by $\{e_\lambda, e_{\lambda^*} = \phi e_\lambda, \xi\}$ and assume that the indices $i,j,k,\cdots$ run over the range $\{1,2,\cdots,m\}$, $\lambda,\mu,\kappa,\cdots$ run over the range $\{1,2,\cdots,q\}$ and $r,s,t,u,\cdots$ take $\{1,2,\cdots,2q\}$. Then we have the following components and relations with respect to the ϕ-basis $\{e_\lambda, e_{\lambda*}, \xi\}$;

$$
(2,5)\qquad
\begin{aligned}
&g_{ji} = \delta_{ji}\\
&\Phi_{\lambda\lambda^*} = -\Phi_{\lambda*\lambda} = 1, \quad \Phi_{j\lambda} = 0 \quad (j \neq \lambda^*), \quad \Phi_{mm} = 0,\\
&K_{kj\lambda\mu^*} + K_{kj\lambda^*\mu} = 0,\\
&K_{\lambda mm\lambda} + K_{\lambda^* mm\lambda^*} = 0,\\
&S_{\lambda^*\mu^*} = S_{\lambda\mu}, \quad S_{\lambda\mu^*} = -S_{\lambda^*\mu}\\
&S_{mm} = 0, \quad S_{\lambda m} = S_{\lambda^* m} = 0,
\end{aligned}
$$

where $\Phi_{\lambda\mu} = \Phi(e_\lambda, e_\mu), K_{\lambda\mu\kappa\tau} = K(e_\lambda, e_\mu, e_\kappa, e_\tau)$ and $S_{\lambda\mu} = S(e_\lambda, e_\mu)$.

3. Theorems

R.L. Bishop and S.I. Goldberg [1] proved

PROPOSITION 3.1. *Let R be a curvature like tensor, i.e., R satisfies*
(1) $R(X,Y,Z,U) = -R(Y,X,Z,U)$,
(2) $R(X,Y,Z,U) = R(Z,U,X,Y)$,
(3) *the first Bianchi's identity.*
Then $R = 0$ if and only if $R(X,Y,X,Y) = 0$ for every orthonormal basis.

In the case of a Kaehlerian manifold, as a special case of Proposition 3.1, T. Kashiwada [3] showed that, for $R = 0$, it suffices that $R_{rsrs} = 0$ for every J-basis $\{e_\lambda, e_{\lambda^*} = Je_\lambda\}$, where J is the almost complex structure. Since (2.4) guarantee that the cosymplectic Bochner curvature tensor B is a curvature like tensor and by means of $\nabla\phi = 0$ and (2.4), we see that

PROPOSITION 3.2. *In an m-dimensional cosymplectic manifold, the cosymplectic Bochner curvature tensor vanishes if and only if $B(e_i, e_j, e_i, e_j) = 0$ for every ϕ-basis.*

In [4], the present author proved

LEMMA 3.3. *Let M be an $m(\geq 9)$-dimensional cosymplectic manifold. The cosymplectic Bochner curvature tensor B of M vanishes if and only if $K_{rstu} = 0 \quad (|r|, |s|, |t|, |u| \neq)$, where writing $|r| = \lambda$ for*

$r = \lambda$ or λ^* *and* $|r|, |s|, |t|, |u| \neq$ *means that* $|r|, |s|, |t|$ *and* $|u|$ *differ from one another.*

Putting

$$B_{kjih} = K_{kjih} + U_{kjih}/(m+1)(m+3),$$

we have

$$B_{rssr} = K_{rssr} + U_{rssr}/(m+1)(m+3),$$

where writing

$$\begin{aligned} U_{rssr} &= -(m+1)(S_{rr} + S_{ss}) + Q \quad (|r| \neq |s|), \\ B_{\lambda\lambda^*\lambda^*\lambda} &= K_{\lambda\lambda^*\lambda^*\lambda} - 8S_{\lambda\lambda}/(m+3) + 4Q/(m+1)(m+3) \end{aligned} \tag{3.1}$$

by use of $H_{\lambda\lambda^*} = S_{\lambda\lambda} = H_{\lambda^*\lambda}$ and (2.5).

THEOREM 3.4. *In an* $m(\geq 9)$*-dimensional cosymplectic manifold* M*, the following relations are equivalent to one another at every point* p *of* M*.*

(1) *The cosymplectic Bochner curvature tensor* $B(p) = 0$.

(2) *For every* ϕ*-basis at* p,

$$A(e_\lambda, e_{\lambda^*}) + A(e_\mu, e_{\mu^*}) = 8(e_\lambda, e_{\lambda^*}) \quad (\lambda \neq \mu),$$

where $A(X,Y)$ *means the sectional curvature with respect to the plane spanned by* X *and* Y*.*

(3) *For each* ϕ*-holomorphic 8-plane* W *in* $T_p(M)$,

$$k_p(W, B) = A(e_1, e_2) + A(e_3, e_4)$$

is independent of ϕ*-basis* $B = \{e_1, \cdots, \phi e_1, \cdots, \phi e_4\}$ *of* W.

(4) *For every orthogonal 8-vectors* $\{e_1, \cdots, e_4, \phi e_1, \cdots, \phi e_4\}$ *of* $T_p(M)$,

$$A(e_1, e_2) + A(e_3, e_4) = A(e_1, e_4) + A(e_2, e_3).$$

Proof. (1) $\longleftrightarrow$ (2) have proved in [4].

(1) $\longrightarrow$ (3) : Assume that $B(p) = 0$, then for a ϕ-basis, it follows

$$K_{rssr} = (S_{rr} + S_{ss})/(m+3) + Q/(m+1)(m+3) \quad (|r|, |s| \neq). \tag{3.2}$$

Let

$$B = \{e_1, e_2, e_3, e_4, \phi e_1, \phi e_2, \phi e_3, \phi e_4\}, \quad \text{and}$$
$$B' = \{e'_1, e'_2, e'_3, e'_4, \phi e'_1, \phi e'_2, \phi e'_3, \phi e'_4\}$$

be bases of $W \subset T_p(M)$. We construct two bases of $T_p(M)$ such that

$$\mathcal{F} = \{e_1, \cdots, e_q, \phi e_1, \cdots, \phi e_q, \xi\}$$
$$\mathcal{F}' = \{e'_1, \cdots, e'_4, e_5, \cdots, e_q, \phi e'_1, \cdots, \phi e'_4, \phi e_5, \cdots, \phi e_q, \xi\},$$

then we have

$$(3.3) \qquad \begin{aligned} A(e_1, e_2) + A(e_3, e_4) = &\{S_{11} + S_{22} + S_{33} + S_{44}\}/(m+3) \\ &+ 2Q/(m+1)(m+3), \end{aligned}$$

by use of (3.2).

Let S_{ii} and S'_{ii} be components of the Ricci curvature with respect to the bases $\mathcal{F}$ and $\mathcal{F}'$, respectively. So, as $Q = \Sigma S_{ii} = \Sigma S'_{ii}$ and $S_{mm} = S'_{mm} = 0$, $S_{\lambda\lambda} = S'_{\lambda\lambda}$ and $S_{\lambda^*\lambda^*} = S_{\lambda^*\lambda^{*\prime}}$ $(\lambda > 4)$, we have $\sum_{i=1}^4 S_{ii} = \sum_{i=1}^4 S'_{ii}$. Hence, by virtue of (3.3), we see that $k_p(W, B)$ is independent of B.

(3) $\longrightarrow$ (4) is trivial.

(4) $\longrightarrow$ (1) : Let $\{e_1, \cdots, e_q, e_1, \cdots, \phi e_q, \xi\}$ be an arbitrary ϕ-basis of $T_p(M)$. For $\{e_\kappa, e_\lambda, e_\mu, e_\nu, \phi e_\kappa, \phi e_\lambda, \phi e_\mu, \phi e_\nu\}$, by assumptions,

$$K_{\kappa\lambda\lambda\kappa} + K_{\mu\nu\nu\mu} = K_{\kappa\nu\nu\kappa} + K_{\lambda\mu\mu\lambda}.$$

We take anothor orthonomal vectors $\{e_\kappa, e'_\lambda, e'_\mu, e_\nu, \phi e_\kappa, \phi e'_\lambda, \phi e'_\mu, \phi e_\nu\}$ such that

$$e'_\lambda = \rho e_\lambda + w e_\mu, \quad e'_\mu = -w e_\lambda + \rho e_\mu \quad (\rho^2 + w^2 = 1, \rho w \neq 0).$$

Since $A(e_\kappa, e'_\lambda) + A(e'_\mu, e_\nu) = A(e_\kappa, e_\nu) + A(e'_\lambda, e'_\mu)$, it follows

$$(3.4) \qquad K_{\lambda\kappa\kappa\mu} = K_{\lambda\nu\nu\mu}$$

and so $g((k(e_\kappa, e'_\lambda)e'_\lambda, e_\nu) = g(K(e_\kappa, e'_\mu)e'_\mu, e_\nu)$ by use of the fact that (3.4) is true for every ϕ-basis. Hence we get $K_{\kappa\lambda\mu\nu} + K_{\kappa\mu\nu\lambda} = 0$ and then $K_{\kappa\mu\lambda\nu} = 0$ by the first Bianchi's identity. Replacing $e_\kappa \to e^*_\kappa$, $e_\lambda \to e^*_\lambda, \cdots$, etc., we obtain $K_{rstu} = 0$ $(|r|, |s|, |t|, |u| \neq)$. Thus, by Lemma 3.3, the cosymplectic Bochner curvature tensor vanishes.

REMARK. It is well known that the cosymplectic Bochner curvature tensor B vanishes in the cosymplectic space form. But the converse is not true, because the locally product manifold $M = M_1(c) \times M_2(-c) \times E^1$ of constant holomorphic sectional curvature $c(c \geq 0)$ and $-c$ with $dimM_1 + dimM_2 = 2q \geq 4$ and $min\{dimM_1, dimM_2\} \geq 2$ is a cosymplectic manifold with vanishing cosymplectic Bochner curvature tensor. But M is not a cosymplectic space form.

References

1. R.L. Bishop and S.I. Goldberg, *Some applications of the generalized Gauss-Bonnet theorem*, Trans. Amer. Math Soc. **112** (1964), 508–535.
2. S.S. Eum, *On the cosymplectic Bochner curvature tensor*, J. Korean Math. Soc. **15** (1978), 29–37
3. T. Kashiwada, *Some characterizations of vanishing Bochner curvature tensor*, Hokkaido Math. J **3** (1974), 290–296.
4. B.H Kim, *Fibred Riemannian spaces with quasi Sasakian structure*, Hiroshima Math. J. **20** (1990), 477–513.
5. G.D. Ludden, *Submanifolds of cosymplectic manifolds*, J Diff Geom. **4** (1970), 237–244.
6. M. Seino, *On vanishing contact Bochner curvature tensor*, Hokkaido Math. J. **9** (1980), 256–267.

Department of Mathematics
Pusan University of Foreign Studies
Pusan 608–738, Korea