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ON Λ–ANNIHILATORS AND PRIME Λ–SEMIMODULES

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In [3], M. Takahashi introduced the notion of Λ-semimodules and studied elementary properties of semimodules. In this paper we study Λ-annihilators and prime Λ-semimodules. First of all, we recall the basic concepts of Λ-semimodules([3], [4]).

A semimodule $A = (A, +, 0)$ consists of a set A, a map $+ : A \times A \to A$ and an element 0 of A such that always

(1) $x + y = y + x$,

(2) $(x + y) + z = x + (y + z)$,

(3) $x + 0 = x$,

for all $x, y, z \in A$.

A subset B of a semimodule A is a sub-semimodule of A if $0 \in B$ and if $x, y \in B$ implies $x + y \in B$.

A semiring with unit 1 $\Lambda = (\Lambda, +, 0, \cdot, 1)$ consists of two data:

(4) $(\Lambda, +, 0)$ is a semimodule,

(5) $(\Lambda, \cdot, 1)$ is a semigroup with 1,

such that always $(\lambda + \mu) \cdot \tau = \lambda \cdot \tau + \mu \cdot \tau$, $\lambda \cdot (\mu + \tau) = \lambda \cdot \mu + \lambda \cdot \tau$ and $0 \cdot \tau = \tau \cdot 0 = 0$. When there is no danger of confusion, we would denote $\lambda \cdot \mu$ by $\lambda\mu$.

Let $\Lambda = (\Lambda, +, 0, \cdot, 1)$ be a semiring. A (left) Λ-semimodule is a semimodule $A = (A, +, 0)$ together with a map $u : \Lambda \times A \to A$ written $u(\lambda, x) = \lambda x$, such that always

(6) $\lambda(x + y) = \lambda x + \lambda y$,

(7) $(\lambda + \mu)x = \lambda x + \mu x$,

(8) $(\lambda\mu)x = \lambda(\mu x)$,

(9) $1x = x$,

(10) $\lambda 0 = 0x = 0$.

A subset B of a (left) Λ-semimodule A is a (left) sub-Λ-semimodule of A if $x, y \in B$ implies $x + y \in B$ and if $\lambda \in \Lambda$, $x \in B$ implies $\lambda x \in B$.

Given two Λ-semimodules A and B, a Λ-homomorphism(or a homomorphism of Λ-semimodules) $u : A \to B$ is a homomorphism of

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semimodules such that $u(\lambda x) = \lambda(u(x))$ for all $x \in A$, $\lambda \in \Lambda$. For a Λ-homomorphism $u : A \to B$, if $u(a) = u(a')$ implies $a = a'$, we say that u is a Λ-monomorphism, while if $b \in B$ implies $b = u(a)$ for some $a \in A$, we say that u is a Λ-epimorphism. Finally u is a Λ-isomorphism if and only if u is both a Λ-monomorphism and a Λ-epimorphism, we denote $u : A \simeq B$.

Let Λ be a semiring with 1. For two Λ-semimodules A and B, consider a Λ-homomorphism $u : A \to B$. Then we define $Ker(u)$, $Im(u)$ and $u(A)$ as follows:

$$Ker(u) = \{a \in A | u(a) = 0\},$$

$$Im(u) = \{b \in B | b + u(a) = u(a') \quad \text{for some} \quad a, a' \in A\}$$

and

$$u(A) = \{b \in B | b = u(a) \quad \text{for some} \quad a \in A\}.$$

$Ker(u)$ is called the kernel of u, and $Im(u)$ is called the image of u, while $u(A)$ is called the proper image of u. Then $Ker(u)$ is a sub-Λ-semimodule of A, and both $Im(u)$ and $u(A)$ are sub-Λ-semimodule of B such that $u(A) \subset Im(u) \subset B$.

1. Λ-annihilators

DEFINITION 1.1. Let Λ be a semiring. A subset I of Λ is a left(right) ideal of Λ if $(I, +, 0)$ is a sub-semimodule of $(\Lambda, +, 0)$ and if $\Lambda I \subset I (I\Lambda \subset I)$. If I is both a left and a right ideal of Λ, I is called an ideal of Λ.

For any semiring Λ, Λ itself and $\{0\}$ are left(right) ideals.

DEFINITION 1.2. Let A be a (left) Λ-semimodule. Then for each $X \subset A$, the (left) Λ-annihilator of X in Λ is

$$l_\Lambda(X) = \{\lambda \in \Lambda | \lambda x = 0 \quad \text{for all} \quad x \in X\},$$

and for each $I \subset \Lambda$, the (right) Λ-annihilator of I in A is

$$r_A(I) = \{a \in A | \lambda a = 0 \quad \text{for all} \quad \lambda \in I\}.$$

THEOREM 1.3. *Let* Λ, Γ *be two semirings and let* ${}_\Lambda A_\Gamma$ *a left* Λ*-right* Γ*-bisemimodule, let* $X \subset A$ *and* $I \subset \Lambda$. *Then*

(11) $l_\Lambda(X)$ *is a left ideal of* Λ.

(12) $r_A(I)$ *is a right sub-*Γ*-semimodule of* A_Γ.

Proof. (11) Obviously $0 \in l_\Lambda(X)$. If $\lambda, \mu \in l_\Lambda(X)$, then $\lambda x = 0$ and $\mu x = 0$ for all $x \in X$. It follows that $(\lambda + \mu)x = \lambda x + \mu x = 0$ for all $x \in X$. Hence $\lambda + \mu \in l_\Lambda(X)$. This shows that $l_\Lambda(X)$ is a sub-semimodule of Λ. Put $\mu \in l_\Lambda(X)$. Then $\mu x = 0$ for all $x \in X$. For any $\lambda \in \Lambda$, $(\lambda\mu)x = \lambda(\mu x) = 0$ for all $x \in X$. Hence $\lambda\mu \in l_\Lambda(X)$. This shows that $\Lambda l_\Lambda(X) \subset l_\Lambda(X)$.

(12) If $a, b \in r_A(I)$, then $\lambda a = 0$ and $\lambda b = 0$ for any $\lambda \in I$. It follows that $\lambda(a + b) = \lambda a + \lambda b = 0$ for all $\lambda \in I$, and hence $a + b \in r_A(I)$. For any $\gamma \in \Gamma$, $\lambda(a\gamma) = (\lambda a)\gamma = 0\gamma = 0$. Hence $a\gamma \in r_A(I)$.

THEOREM 1.4. *If* $u : A \to B$ *is a homomorphism of* Λ*-semimodules such that* $Im(u) = B$, *then* $l_\Lambda(A) \subset l_\Lambda(B)$.

Proof. If $\lambda \in l_\Lambda(A)$, then $\lambda a = 0$ for all $a \in A$. Since $Im(u) = B$, for any $b \in B$, there exist $a, a' \in A$ such that $b + u(a) = u(a')$. Therefore $\lambda b + \lambda u(a) = \lambda u(a')$ for all $b \in B$. Since $\lambda u(a) = u(\lambda a) = 0$ and $\lambda u(a') = u(\lambda a') = 0$, we have $\lambda b = 0$ and hence $\lambda \in l_\Lambda(B)$.

THEOREM 1.5. *Let* A *be a (left)* Λ*-semimodule and let* X, Y *be subsets of* A *and* I, J *subsets of* Λ. *Then we have the followings:*

(13) $X \subset Y$ *implies* $l_\Lambda(X) \supset l_\Lambda(Y)$,

(14) $I \subset J$ *implies* $r_A(I) \supset r_A(J)$,

(15) $X \subset r_A l_\Lambda(X)$,

(16) $I \subset l_\Lambda r_A(I)$,

(17) $l_\Lambda(X) = l_\Lambda r_A l_\Lambda(X)$,

(18) $r_A(I) = r_A l_\Lambda r_A(I)$.

Proof. (13 - 16) are easy. (17) and (18) follow from (13 - 16).

2. Prime Λ-semimodules

DEFINITION 2.1. A Λ-semimodule A is called prime provided $A \neq 0$ and $l_\Lambda(M) = l_\Lambda(A)$ for all nonzero sub-Λ-semimodule M of A.

THEOREM 2.2. *Every nonzero sub-Λ-semimodule of a prime Λ-semimodule is prime.*

Proof. Let A be a prime Λ-semimodule and let M be any nonzero sub-Λ-semimodule of A. Then we have $l_\Lambda(M) = l_\Lambda(A)$. If N is any nonzero sub-Λ-semimodule of M, then N is also a sub-Λ-semimodule of A, and $l_\Lambda(N) = l_\Lambda(A)$. Therefore we get $l_\Lambda(N) = l_\Lambda(M)$, it follows that M is prime.

By above Theorem 2.2, we have the following corollaries.

COROLLARY 2.3. *Let $u : A \to B$ be a nonzero Λ-homomorphism. If B is a prime Λ-semimodule, then $Im(u)$ and $u(A)$ are primes.*

COROLLARY 2.4. *Let $u : A \to B$ be a Λ-homomorphism. If A is a prime Λ-semimodule, then $Ker(u)$ is prime.*

COROLLARY 2.5. *Let $u : A \to B$ be a Λ-homomorphism. Then we have*

(19) *The sequence $0 \longrightarrow Coim(u) \xrightarrow{u_*} Im(u) \longrightarrow 0$ is exact.*

(20) *If u is k-regular and if B is a prime Λ-semimodule, then $Coim(u)$ is prime.*

Proof. (19). See [3].

(20). Since u is k-regular if and only if u_* is injective if and only if $Coim(u) \simeq u(A)$, we have the result by Corollary 2.3.

THEOREM 2.6. *If A is a Λ-semimodule such that $l_\Lambda(A) = 0$, then A is a prime Λ-semimodule.*

Proof. Since $l_\Lambda(A) = 0$, we have $l_\Lambda(M) = l_\Lambda(A)$ for any nonzero sub-Λ-semimodule M of A. Therefore A is prime.

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