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A NOTE ON FUZZY SUBALGEBRAS OF A BCK/BCI–ALGEBRA

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This note is continuation of our study [4]. We have proved in [4] that any subalgebra of a BCK/BCI-algebra X can be realized as a level subalgebra of some fuzzy subalgebra of X.

In this note we prove a generalization of this result.

We review some definitions and results. We refer the reader to [1], [4], [5] and [6] for details.

DEFINITION 1. Let X be a set. A fuzzy set in X is a mapping $\mu : X \to [0,1]$.

DEFINITION 2. Let X be a BCK/BCI-algebra. A fuzzy set μ in X is called a fuzzy subalgebra of X if, for all $x, y \in X$,

$$\mu(x * y) \geq \min(\mu(x), \mu(y)).$$

DEFINITION 3. Let μ be a fuzzy set in a set X. For $t \in [0,1]$, the set

$$\mu_t := \{x \in X : \mu(x) \geq t\}$$

is called a level subset of μ.

THEOREM 4. [4] *Let X be a BCK/BCI-algebra and let μ be a fuzzy set in X such that μ_t is a subalgebra of X for all $t \in [0,1]$, $t \leq \mu(0)$. Then μ is a fuzzy subalgebra of X.*

DEFINITION 5. [4] Let X be a BCK/BCI-algebra and let μ be a fuzzy subalgebra of X. The subalgebras μ_t, $t \in [0,1]$ and $t \leq \mu(0)$, are called level subalgebras of μ.

THEOREM 6. [4] *Any subalgebra of a BCK/BCI-algebra X can be realized as a level subalgebra of some fuzzy subalgebra of X.*

As a generalization of Theorem 6, we prove the following theorem.

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THEOREM 7. *Let X be a BCK/BCI-algebra. Then given any chain of subalgebras*

$$A_0 \subset A_1 \subset \cdots \subset A_r = X,$$

there exists a fuzzy subalgebra of X whose level subalgebras are exactly the subalgebras of this chain.

Proof. Consider a set of numbers

$$t_0 > t_1 > ... > t_r,$$

where each t_i is in $[0,1]$. Let $\mu : X \to [0,1]$ be a fuzzy set defined by

$$\mu(A_0) = t_0 \quad \text{and} \quad \mu(A_i - A_{i-1}) = t_i, 0 < i \leq r.$$

We claim that μ is a fuzzy subalgebra of X. Let $x, y \in X$. Then we distinguish two cases as follows:

Case 1. $x, y \in A_i - A_{i-1}$. Then by definition of μ,

$$\mu(x) = t_i = \mu(y).$$

Since A_i is a subalgebra, it follows that $x * y \in A_i$, and so either $x * y \in A_i - A_{i-1}$ or $x * y \in A_{i-1}$. In any case we conclude that

$$\mu(x * y) \geq t_i = \min(\mu(x), \mu(y)).$$

Case 2. For $i > j$, $x \in A_i - A_{i-1}$ and $y \in A_j - A_{j-1}$. Then

$$\mu(x) = t_i, \mu(y) = t_j, \quad \text{and} \quad x * y \in A_i$$

because A_i is a subalgebra and $A_j \subset A_i$. It follows that

$$\mu(x * y) \geq t_i = \min(\mu(x), \mu(y)).$$

Hence we know that μ is a fuzzy subalgebra of X. From the definition of μ, it follows that

$$\mathrm{Im}(\mu) = \{t_0, t_1, ..., t_r\}.$$

Thus the level subalgebras of μ are given by the chain of subalgebras

$$\mu_{t_0} \subset \mu_{t_1} \subset ... \subset \mu_{t_r} = X.$$

Now $\mu_{t_0} = \{x \in X : \mu(x) \geq t_0\} = A_0$. Finally we prove that $\mu_{t_i} = A_i$ for $0 < i \leq r$. Clearly $A_i \subseteq \mu_{t_i}$. If $x \in \mu_{t_i}$, then $\mu(x) \geq t_i$ which implies that $x \notin A_j$ for $j > i$. Hence $\mu(x) \in \{t_1, t_2, ..., t_i\}$, and so $x \in A_k$ for some $k \leq i$. As $A_k \subseteq A_i$, it follows that $x \in A_i$. Therefore we obtain $\mu_{t_i} = A_i$ for $0 \leq i \leq r$. This completes the proof.

REMARK. In Theorem 7, we have shown the existence of a fuzzy subalgebra whose level subalgebras are the subalgebras of the given finite chain. However, from the proof of the theorem it is clear that such a fuzzy subalgebra cannot be unique. In fact, we have given an example in [4] of the fact that two distinct fuzzy subalgebras of a finite BCK/BCI-algebra may have the identical family of level subalgebras.

Further, we have shown in [4; Theorem 15] that for a finite BCK/BCI-algebra X, two fuzzy subalgebras with the identical family of level subalgebras are equal if and only if their image sets are equal.

In [4] we introduced an equivalence relation for the family $\mathcal{F}$ of all fuzzy subalgebras of a finite BCK/BCI-algebra X given by

$$\mu \sim \nu$$

if and only if the fuzzy subalgebras μ and ν have the identical family of level subalgebras. This equivalence partitions $\mathcal{F}$ into equivalence classes. We have shown in [4; Theorem 17] that if X is a finite BCK/BCI-algebra then the number of distinct equivalence classes in $\mathcal{F}$ is finite.

Now let A be a subalgebra of a finite BCK/BCI-algebra X. Let $n(A)$ denote the number of chains of subalgebras of X, ending in X but not necessarily beginning with the trivial subalgebra $\{0\}$, in which A is a member. Then clearly $n(A)$ is a nonzero positive integer. From the Theorem 7, we have the following corollary:

COROLLARY 8. *If A is a subalgebra of a finite BCK/BCI-algebra X, then $n(A)$ is equal to the number of equivalence classes of fuzzy subalgebras of X such that A is a level subalgebra of any member of the equivalence class.*

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