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CHARACTERIZATIONS ON KL-PRODUCT BCI-ALGEBRAS

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The first author of this note and X. L. Xin introduced the concept of KL-product BCI-algebras and gave some elementary properties ([4]). Now we continue to study these algebras. Let us recall some definitions and results, which are necessary for development of this paper.

An algebra $(X;*,0)$ of type (2, 0) is said to be a BCI-algebra if it satisfies the following conditions:

BCI-1 $(x*y)*(x*z) \leq z*y$,
BCI-2 $x*(x*y) \leq y$,
BCI-3 $x \leq x$,
BCI-4 $x \leq y$ and $y \leq x$ imply $x = y$,
BCI-5 $x \leq y$ if and only if $x*y = 0$.

The following identities hold for any BCI-algebra X:

(1) $x*0 = x$,
(2) $(x*y)*z = (x*z)*y$,
(3) $x*(x*(x*y)) = x*y$,
(4) $0*(x*y) = (0*x)*(0*y)$.

The above definition and properties can be found in [1] and [4].

DEFINITION 1. ([4]) Suppose $(X;*,0)$ is a BCI-algebra. If there are a BCK-algebra $(Y;*_1,0_1)$ and a p-semisimple BCI-algebra $(Z;*_2,0_2)$ such that $X \cong Y \times Z$, then $(X;*,0)$ is said to be a KL-product BCI-algebra.

DEFINITION 2. ([3]) An element a of a BCI-algebra X is said to be an atom if, for all x in X, $x*a = 0$ implies $x = a$. The set of all atoms of X is denoted by $L(X)$. For any atom a, $V(a) = \{x \in X : a \leq x\}$ is called a branch of X.

Obviously, $V(0)$ is the BCK-part of X and denoted by $B(X)$. For details of atoms and branchs we refer readers to [3].

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DEFINITION 3. ([1]) A nonempty subset I of a BCI-algebra X is said to be an ideal if it satisfies:

(5) $0 \in I$,

(6) $x * y \in I$ and $y \in I$ imply $x \in I$.

PROPOSITION 4. ([4]) *For X a BCI-algebra, the following conditions are equivalent:*

(7) *X is of KL-product,*

(8) *$L(X)$ is an ideal of X,*

(9) *$x * a = y * a$ implies $x = y$ for any a in $L(X)$.*

Next we give other characterizations of KL-product BCI-algebras.

THEOREM 5. *A BCI-algebra $(X, *, 0)$ is of KL-product if and only if, for any $x \in X$ and for any $b \in L(X)$, we have*

(10) $x = (x * b) * (0 * b)$.

Proof. ($\Rightarrow$) If b is an atom of X, then by [3](13),

$$\begin{aligned}&(x * ((x * b) * (0 * b))) * b \\ &= (x * b) * ((x * b) * (0 * b)) \\ &= 0 * b.\end{aligned}$$

It follows from (9) that

$$x * ((x * b) * (0 * b)) = 0.$$

On the other hand,

$$\begin{aligned}&((x * b) * (0 * b)) * x \\ &= ((x * x) * b) * (0 * b) \\ &= (0 * b) * (0 * b) \\ &= 0.\end{aligned}$$

Hence $x = (x * b) * (0 * b)$, i.e., (10) holds.

($\Leftarrow$) Suppose $x = (x*b)*(0*b)$ for any $x \in X$ and for any $b \in L(X)$. We now prove that $L(X)$ is an ideal of X. Assume that $x * b \in L(X)$ and $b \in L(X)$. Denoted $a = 0*(0*x) \in L(X)$, we have $x*b \in V(a*b)$ by [3](16). Thus $x * b = a * b$, and so

$$x = (x * b) * (0 * b) = (a * b) * (0 * b) \in L(X).$$

This means that $L(X)$ is an ideal of X. This completes the proof.

THEOREM 6. *A BCI-algebra X is of KL-product if and only if, for any $x, y \in X$ and for any $a, b \in L(X)$,*
(11) $(x * a) * (y * b) = (x * y) * (a * b)$.

Proof. Suppose X is of KL-product. Since

$$\begin{aligned} &(((x * y) * (a * b)) * ((x * a) * (y * b))) * a \\ &= (((x * a) * ((x * a) * (y * b))) * y) * (a * b) \\ &\leq ((y * b) * y) * (a * b) \\ &= (0 * b) * (a * b) \\ &\leq 0 * a, \end{aligned}$$

noticing that $0 * a \in L(X)$ we have

$$(((x * y) * (a * b)) * ((x * a) * (y * b))) * a = 0 * a.$$

It follows from (9) that
(12) $((x * y) * (a * b)) * ((x * a) * (y * b)) = 0$.
Because

$$\begin{aligned} &(((x * a) * (y * b)) * ((x * y) * (a * b))) * (a * b) \\ &= (((x * (a * b)) * ((x * y) * (a * b))) * (y * b)) * a \\ &\leq ((x * (x * y)) * (y * b)) * a \\ &\leq (y * (y * b)) * a \\ &\leq b * a \\ &= 0 * (a * b), \qquad \text{[by [3](11)]} \end{aligned}$$

we obtain

$$(((x * a) * (y * b)) * ((x * y) * (a * b))) * (a * b) = 0 * (a * b).$$

Using (9) the following identity holds
(13) $((x * a) * (y * b)) * ((x * y) * (a * b)) = 0$.
Combining (12) and (13) we obtain (11).

Conversely, suppose that (11) holds. If, for $a \in L(X)$, we have $x * a = y * a$, then

$$x * y = (x * y) * (a * a) = (x * a) * (y * a) = 0.$$

Likewise we have that $y * x = 0$, and so $x = y$. This says that (9) holds. By Proposition 4, X is of KL-product. The proof is completed.

To be motivated by this theorem, we introduce a mapping as follows.

DEFINITION 7. Suppose $(X, *, 0)$ is a BCI-algebra. The mapping $p : X \to X$ is defined by putting $p(x) = x * a$ for all $x \in X$, where $a = 0 * (0 * x) \in L(X)$.

By the necessity of Theorem 6 we have

THEOREM 8. *If X is a KL-product BCI-algebra, then p is an endomorphism on X.*

Open problem. Does the inverse of Theorem 8 hold?

THEOREM 9. *A BCI-algebra $(X, *, 0)$ is of KL-product if and only if there exists an endomorphism f on X such that for any $a \in L(X)$, $f|_{V(a)}$, the restriction of f to $V(a)$, is a bijection from $V(a)$ onto $B(X)$.*

Proof. Suppose X is of KL-product. By Theorem 8, the mapping $p : X \to X$ is an endomorphism and $Im(p) = B(X)$. Now it suffices to show that for any $a \in L(X)$, $p|_{V(a)}$ is a bijection. If $x, y \in V(a)$ with $x \neq y$, then $x * y \neq 0$ or $y * x \neq 0$. Since by (11)

$$\begin{aligned} p|_{V(a)}(x) * p|_{V(a)}(y) &= p(x) * p(y) \\ &= (x * a) * (y * a) \\ &= (x * y) * (a * a) \\ &= x * y, \end{aligned}$$

it follows that $p(x) * p(y) \neq 0$ or $p(y) * p(x) \neq 0$. Hence $p|_{V(a)}$ is an injection from $V(a)$ to $B(X)$.

By [3](16) we have $x * (0 * a) \in V(a)$ for any $x \in B(X)$, and so by Theorem 5 the following holds:

$$p|_{V(a)}(x * (0 * a)) = (x * (0 * a)) * a = (x * a) * (0 * a) = x.$$

This says that $p|_{V(a)}$ is a surjection from $V(a)$ to $B(X)$. Hence it is a bijection from $V(a)$ onto $B(X)$.

Conversely, suppose that there is an endomorphism f on X such that for any $a \in L(X)$, $f|_{V(a)}$ is a bijection from $V(a)$ to $B(X)$. It is easy to verify that for any $a \in L(X)$, $f(a) = 0$. Hence for any $x \in X$ and for any $b \in L(X)$,

$$\begin{aligned} & f((x * b) * (0 * b)) \\ &= (f(x) * f(b)) * (f(0) * f(b)) \\ &= (f(x) * 0) * (0 * 0) \\ &= f(x). \end{aligned}$$

Since x and $(x * b) * (0 * b)$ are in the same branch, we obtain $x = (x * b) * (0 * b)$. By Theorem 5, X is of KL-product. The proof is completed.

References

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