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WEAK CONVERGENCE TO FIXED POINTS OF ALMOST–ORBITS OF NONLIPSCHITZIAN SEMIGROUPS

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1. Introduction

Let G be a semitopological semigroup, i.e., G is a semigroup with a Hausdorff topology such that for each $a \in G$ the mappings $s \mapsto a \cdot s$ and $s \mapsto s \cdot a$ from G to G are continuous. G is called right reversible if any two closed left ideals of G have nonvoid intersection. In this case, $(G, \succeq)$ is a directed system when the binary relation " $\succeq$" on G is defined by

$$t \succeq s \quad \text{if and only if} \quad \{s\} \cup \overline{Gs} \supseteq \{t\} \cup \overline{Gt}, \quad s, t \in G.$$

Right reversible semitopological semigroups include all commutative semigroups and all semitopological semigroups which are right amenable as discrete semigroups (see [4, p.335]). Left reversibility of G is defined similarly. G is called reversible if it is both left and right reversible.

Let G be a semitopological semigroup with a binary relation "$\succeq$" which directs G. Let C be a nonempty closed convex subset of a real Banach space E and let a family $\Im = \{S(t) : t \in G\}$ be a (continuous) representation of G as continuous mappings on C into C, i.e., $S(ts)x = S(t)S(s)x$ for all $t, s \in G$ and $x \in C$, and for every $x \in C$, the mapping $t \mapsto S(t)x$ from G into C is continuous. In this paper, we also consider a non-Lipschitzian semigroup of continuous mappings : a representation $\Im = \{S(t) : t \in G\}$ of G on C is said to be a semigroup of asymptotically nonexpansive type (simply, a.n.t.) on C if, for each $x \in C$,

$$\inf_{s \in G} \sup_{t \succeq s} \sup_{y \in C} \big(\|S(t)x - S(t)y\| - \|x - y\|\big) \leq 0.$$

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Immediately, we can see that the semigroups of a.n.t. include all semigroups of nonexpansive mappings with directed systems. $\Im = \{S(t) : t \in G\}$ is called reversible [resp., right(left) reversible] if G is reversible [resp., right(left) reversible]. For a mapping $S : C \to C$, we define $S(n) = S^n$ for each $n \in G = N$, where N denotes the set of natural numbers. Then, when the semigroup $\Im = \{S(n) : n \in G\}$ is of a.n.t., the mapping $S : C \to C$ is simply said to be of a.n.t. In particular, if $\Im = \{S(t) : t \in G\}$ is a Lipschitzian representation of G with an additional condition, i.e., $\limsup_t k_t \leq 1$ (see [7]), and if C is bounded, then it is obviously of a.n.t. And we say that a function $u : G \to C$ is an almost-orbit of $\Im = \{S(t) : t \in G\}$ (see [1], [7]) if G is right reversible and

$$\lim_t \Big(\sup_{s \in G} \|u(st) - S(s)u(t)\| \Big) = 0.$$

In [7], Takahashi-Zhang established the weak convergence of an almost-orbit of a noncommutative Lipschitzian semigroup in a Banach space. And in [6], Lau-Takahashi proved the nonlinear ergodic theorems for a noncommutative nonexpansive semigroup in the space. In this paper, we shall establish the weak convergence to a fixed point of an almost-orbit $\{u(t) : t \in G\}$ of the right reversible semigroup $\Im = \{S(t) : t \in G\}$ of a.n.t. in a uniformly convex Banach space with a Fréchet differentiable norm, which extends the result according to Takahashi-Zhang [7].

2. Preliminaries and some lemmas

Let C be a nonempty closed convex subset of a real Banach space E and let G be a semitopological semigroup with a binary relation "$\succeq$" which directs G. A family $\Im = \{S(t) : t \in G\}$ of continuous mappings from C into itself is said to be a (continuous) representation of G on C if $\Im$ satisfies the following:

(a) $S(ts)x = S(t)S(s)x$ for all $t, s \in G$ and $x \in C$;

(b) for every $x \in C$, the mapping $t \mapsto S(t)x$ from G into C is continuous. A representation $\Im = \{S(t) : t \in G\}$ of G on C is said to be a semigroup of asymptotically nonexpansive type (simply, a.n.t.) on C if, for each $x \in C$,

$$\inf_{s \in G} \sup_{t \succeq s} \sup_{y \in C} \big(\|S(t)x - S(t)y\| - \|x - y\| \big) \leq 0. \tag{2.1}$$

Immediately, we can see that the semigroups of a.n.t. indude all semigroups of nonexpansive mappings with directed systems. In particular, if $\Im = \{S(t) : t \in G\}$ is a Lipschitzian representation of G with an additional condition, i.e., $\limsup_t k_t \leq 1$ (see [7]), and if C is bounded, then it is obviously of a.n.t.

Let G be a semitopological semigroup, i.e., G is a semigroup with a Hausdorff topology such that for each $a \in G$ the mappings $s \mapsto a \cdot s$ and $s \mapsto s \cdot a$ from G to G are continuous. G is called right reversible if any two closed left ideals of G have nonvoid intersection. In this case, $(G, \succeq)$ is a directed system when the binary relation " $\succeq$" on G is defined by

$$t \succeq s \quad \text{if and only if} \quad \{s\} \cup \overline{Gs} \supseteq \{t\} \cup \overline{Gt}, \quad s,t \in G.$$

Right reversible semitopological semigroups include all commutative semigroups and all semitopological semigroups which are right amenable as discrete semigroups (see [4, p.335]). Left reversibility of G is defined similarly. G is called reversible if it is both left and right reversible.

In particular, if G is right reversible, and if $\Im = \{S(t) : t \in G\}$ is a semigroup of a.n.t. on C, then A function $u : G \to C$ is called an almost-orbit of $\Im = \{S(t) : t \in G\}$ if

$$\lim_t (\sup_s \|u(st) - S(s)u(t)\| \ ) = 0. \tag{2.2}$$

With each $x \in E$, we associate the set

$$J(x) = \{x^* \in E^* : < x, x^* >= \|x\|^2 = \|x^*\|^2\}. \tag{2.3}$$

Using the Hahn-Banach theorem it is immediately clear that $J(x) \neq \emptyset$ for any $x \in E$. The multivalued operator $J : E \to 2^{X^*}$ is called the duality mapping of E. The norm of E is said to be Gâteaux diffentiable (and E is said to be smooth) if for each $x, y \in S$,

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.4}$$

exists, where S denotes the unit sphere of E. It is said to be Fréchet differentiable if, for each $x \in S$, this limit (2.4) is attained uniformly for

$y \in S$. Then it is well-known that if E is smooth, the duality mapping J is single-valued. We also know that if E has a Fréchet differentiable norm, then J is norm to norm continuous.

In order to measure the degree of strict convexity (rotundity) of E, we define its modulus of convexity $\delta : [0,2] \to [0,1]$ by

$$(2.5)\quad \delta(\varepsilon) = \inf\{1 - \frac{1}{2}\|x+y\| : \|x\| \le 1,\ \|y\| \le 1,\ \text{and}\ \|x-y\| \ge \varepsilon\}.$$

The characteristic of convexity $\varepsilon_\circ$ of E is also defined by

$$(2.6)\qquad \varepsilon_\circ = \varepsilon_\circ(E) = \sup\{\varepsilon : \delta(\varepsilon) = 0\}.$$

It is well-known (see [3]) that the modulus of convexity δ satisfies the following properties:

$$(2.7)\quad \begin{cases} \text{(a) } \delta \text{ is increasing on } [0,2], \text{ and moreover strictly} \\ \qquad \text{increasing on } [\varepsilon_\circ, 2]; \\ \text{(b) } \delta \text{ is continuous on } [0,2) \text{ (but not necessarily at } \varepsilon = 2); \\ \text{(c) } \delta(2) = 1 \text{ if and only if } E \text{ is strictly convex;} \\ \text{(d) } \delta(0) = 0 \text{ and } \lim_{\varepsilon \to 2-} \delta(\varepsilon) = 1 - \frac{1}{2}\varepsilon_\circ; \\ \text{(e) } \|a - x\| \le r, \|a - y\| \le r \text{ and } \|x - y\| \ge \varepsilon \\ \qquad \Longrightarrow \|a - \frac{1}{2}(x+y)\| \le r(1 - \delta(\varepsilon/r)). \end{cases}$$

A Banach space E is said to be uniformly convex if $\delta(\varepsilon) > 0$ for all positive ε; equivalently $\varepsilon_\circ = 0$. Obviously, any uniformly convex space is both strictly convex and reflexive. By properties above, we can see that if E is uniformly convex, then δ is strictly increasing and continuous on [0,2] (see [2]).

It is easy that if G is right reversible and $u = \{u(t) : t \in G\}$ is an almost-orbit of the semigroup $\Im = \{S(t) : t \in G\}$ of a.n.t., then $F(\Im) \subseteq E(u)$, where $E(u) = \{y \in C : \lim_t \|u(t) - y\| \text{ exists}\}$ and $F(\Im)$ denotes the set of all common fixed points of $\Im$.

Lemma 2.1. *Let C be a nonempty closed convex of a uniformly convex Banach space E. Let G be right reversible and let $\Im = \{S(t) : t \in G\}$ be a semigroup of a.n.t. on C. Let $u = \{u(t) : t \in G\}$ be an almost-orbit of $\Im$. Suppose $F(\Im) \neq \emptyset$ and let $y \in F(\Im)$ and $0 < \alpha \le \beta < 1$. Then, for any $\varepsilon > 0$, there is $t_\circ \in G$ such that*

$$\|S(t)(\lambda u(s) + (1-\lambda)y) - (\lambda S(t)u(s) + (1-\lambda)y)\| < \varepsilon$$

for all $t, s \succeq t_\circ$ *and* $\lambda \in [\alpha, \beta]$.

Proof. Let $\varepsilon > 0, c = \min\{2\lambda(1-\lambda) : \alpha \leq \lambda \leq \beta\}$ and let $r = \lim_t \|u(t) - y\|$. If $r = 0$, since $\Im = \{S(t) : t \in G\}$ is of a.n.t. on C, there exists $t_\circ \in G$ such that

$$\|y - S(t)z\| < \|y - z\| + \frac{\varepsilon}{4}$$

and

$$\|u(t) - y\| < \frac{\varepsilon}{4} \quad \text{for } t \succeq t_\circ \text{ and } z \in C.$$

Hence, for $s, t \succeq t_\circ$ and $0 \leq \lambda \leq 1$,

$$\begin{aligned}
&\big\|S(t)\big(\lambda u(s) + (1-\lambda)y\big) - \big(\lambda S(t)u(s) + (1-\lambda)y\big)\big\| \\
\leq &\big\|S(t)\big(\lambda u(s) + (1-\lambda)y\big) - y\big\| + \lambda\|S(t)u(s) - y\| \\
\leq &2\big(\lambda\|u(s) - y\| + \frac{\varepsilon}{4}\big) < \varepsilon.
\end{aligned}$$

Now, let $r > 0$. Then we can choose $d > 0$ so small that

$$(r + d)\Big[1 - c\delta(\frac{\varepsilon}{r + d})\Big] = r_\circ < r,$$

where δ is the modulus of convexity of E. On taking $a > 0$ with $a < \min\{\frac{d}{2}, \frac{r - r_\circ}{2}\}$, there exists $t_\circ \in G$ such that

$$r - a < \|u(t) - y\| < r + a, \tag{2.8}$$

$$\|y - S(t)z\| < \|y - z\| + \frac{c}{4}d, \tag{2.9}$$

and

$$\|u(st) - S(s)u(t)\| \leq a, \tag{2.10}$$

for all $t \succeq t_\circ$, $s \in G$ and $z \in C$. Suppose that

$$\big\|S(t)\big(\lambda u(s) + (1-\lambda)y\big) - \big(\lambda S(t)u(s) + (1-\lambda)y\big)\big\| \geq \varepsilon$$

for some $s,t \succeq t_\circ$ and $\lambda \in [\alpha,\beta]$. Put $z = \lambda u(s) + (1-\lambda)y$, $u = (1-\lambda)(S(t)z - y)$ and $v = \lambda(S(t)u(s) - S(t)z)$. Then, by (2.8) and (2.9), we have

$$\begin{aligned}\|u\| &\le (1-\lambda)\big(\|y-z\| + \frac{c}{4}d\big)\\ &= (1-\lambda)\big(\lambda\|u(s)-y\| + \frac{c}{4}d\big)\\ &< (1-\lambda)\big(\lambda(r+\frac{d}{2}) + \frac{c}{4}d\big)\\ &< \lambda(1-\lambda)(r+d)\end{aligned}$$

and

$$\|v\| < \lambda(1-\lambda)(r+d).$$

We also have that

$$\|u-v\| = \|S(t)z - \big(\lambda S(t)u(s) + (1-\lambda)y\big)\| \ge \varepsilon$$

and $\lambda u + (1-\lambda)v = \lambda(1-\lambda)\big(S(t)u(s) - y\big)$. So, by uniform convexity of E, we have

$$\begin{aligned}\lambda(1-\lambda)\|S(t)u(s)-y\| &= \|\lambda u + (1-\lambda)v\|\\ &\le \lambda(1-\lambda)(r+d)\Big[1 - 2\lambda(1-\lambda)\delta(\frac{\varepsilon}{r+d})\Big]\\ &\le \lambda(1-\lambda)r_\circ,\end{aligned}$$

and hence $\|S(t)u(s) - y\| \le r_\circ$. Then, it follows from (2.10) that

$$\begin{aligned}\|u(ts)-y\| &\le \|u(ts) - S(t)u(s)\| + \|S(t)u(s)-y\|\\ &< a + r_\circ < r - a,\end{aligned}$$

which contradicts to (2.8) and the proof is complete.

For $x,y \in E$, we denote by $[x,y]$ the set $\{\lambda x + (1-\lambda)y : 0 \le \lambda \le 1\}$. The following lemma was proved by Lau-Takahashi [6, Lemma 3].

LEMMA 2.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E with a Fréchet differentiable norm and let $\{x_\alpha\}$ be a bounded net in C. Let $z \in \bigcap_\beta \overline{co}\{x_\alpha : \alpha \succeq \beta\}$, $y \in C$ and $\{y_\alpha\}$ a net of elements in C with $y_\alpha \in [y, x_\alpha]$ and*

$$\|y_\alpha - z\| = \min\{\ \|u-z\| : u \in [y,x_\alpha]\}.$$

If $y_\alpha \to y$, then $y = z$.

3. Weak convergence theorem

In this section, we study the weak convergence of an almost-orbit $\{u(t) : t \in G\}$ of $\Im = \{S(t) : t \in G\}$ in a uniformly convex Banach space E with a Fréchet differentiable norm. By using Lemma 2.1 and Lemma 2.2, we obtain the similar result as Theorem 2 of [6] for the semigroup $\Im = \{S(t) : t \in G\}$ of asymptotically nonexpansive type.

THEOREM 3.1. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty closed convex subset of E. Let G be right reversible and let $\Im = \{S(t) : t \in G\}$ be a semigroup of a.n.t. on C. Suppose that $u = \{u(t) : t \in G\}$ is an almost-orbit of $\Im$ and $F(\Im) \neq \emptyset$. Then the set $\bigcap_s \overline{co}\{u(t) : t \succeq s\} \cap F(\Im)$ consists of at most one point.*

Proof. Let $W(u) = \bigcap_s \overline{co}\{u(t) : t \succeq s\}$. Suppose that $f, g \in W(u) \cap F(\Im)$ and $f \neq g$. Put $h = (f+g)/2$ and $r = \lim_s \|u(s) - g\|$ by Lemma 2.1. Since $h \in W(u)$, we have $\|h - g\| \leq r$. For each $s \in G$, choose $p_s \in [u(s), h]$ such that

$$\|p_s - g\| = \min\{\|y - g\| : y \in [u(s), h]\}.$$

By the definition of p_s, we have $\|p_s - g\| \leq \|(p_s + h)/2 - g\| \leq \|h - g\|$ for all $s \in G$. If $\liminf_s \|p_s - g\| = \|h - g\|$, then $\{p_s\}$ converges strongly to h. Hence, by Lemma 2.2, we have $h = g$. This contradicts $f \neq g$. To complete the proof, we suppose that

$$\liminf_s \|p_s - g\| < \|h - g\|.$$

Then there exist $c > 0$ and $t_\alpha \in G$ such that $t_\alpha \succeq \alpha$ and

$$\|p_{t_\alpha} - g\| + c < \|h - g\| \qquad \text{for every } \alpha \in G.$$

Put $p_{t_\alpha} = a_\alpha u(t_\alpha) + (1 - a_\alpha)h$ for every α. Then there is $\beta > 0$ and $\gamma < 1$ such that $\beta \leq a_\alpha \leq \gamma$ for every α. By (2.1),(2.2), and Lemma 2.1, there exists $\alpha_o \in G$ such that

$$\|g - S(s)z\| < \frac{c}{3} + \|g - z\|,$$

$$\|u(st) - S(s)u(t)\| < \frac{c}{3},$$

and

$$\|S(s)(\lambda u(t)+(1-\lambda)h)-(\lambda S(s)u(t)+(1-\lambda)h)\|<\frac{c}{3},$$

for all $s,t\succeq\alpha_o$, $z\in C$ and $\lambda\in[\beta,\gamma]$. For $s\succeq\alpha_o$, since $t_{\alpha_o}\succeq\alpha_o$, the above inequalities imply that

$$\begin{aligned}\|p_{st_{\alpha_o}}-g\|\leq&\|a_{\alpha_o}u(st_{\alpha_o})+(1-a_{\alpha_o})h-g\|\\ \leq&a_{\alpha_o}\|u(st_{\alpha_o})-S(s)u(t_{\alpha_o})\|+\\ &\|S(s)p_{t_{\alpha_o}}-\big(a_{\alpha_o}S(s)u(t_{\alpha_o})+(1-a_{\alpha_o})h\big)\|+\|S(s)p_{t_{\alpha_o}}-g\|\\ <&\|h-g\|.\end{aligned}$$

Let $\beta_o=\alpha_o t_{\alpha_o}$ and $t\succeq\beta_o$. Then,since G is right reversible, $t\in\{\beta_o\}\cup\overline{G\beta_o}$, we may assume $t\in\overline{G\beta_o}$. Let $\{t_\beta\}$ be a net in G such that $t_\beta\beta_o\to t$. Then, $t=st_{\alpha_o}$, $s=\lim_\beta t_\beta\alpha_o\in\overline{G\alpha_o}$ and hence $s\succeq\alpha_o$. Therefore, we obtain $\|p_t-g\|<\|h-g\|$ for all $t\succeq\beta_o$. So, we have $p_t\neq h$ for all $t\succeq\beta_o$. Now let $t\succeq\beta_o$ and $u_k=k(h-p_t)+p_t$ for all $k\geq 1$. Then $\|u_k-g\|\geq\|h-g\|$ for all $k\geq 1$ and hence $\langle h-u_k,J(g-h)\rangle\geq 0$ for all $k\geq 1$, where J is the duality mapping of X and $\langle x,f\rangle$ denotes the value of $f\in X^*$ at $x\in X$. Then, since $p_t\in[u(t),h]$, it easily follows that $\langle h-u(t),J(g-h)\rangle\leq 0$ for all $t\succeq\beta_o$. Immediately, we obtain $\langle h-y,J(g-h)\rangle\leq 0$ for all $y\in\overline{co}\{u(t):t\succeq\beta_o\}$, and hence $h=g$. This contradicts $f\neq g$ and so the proof is complete.

As a direct consequence, we present the following weak convergence of an almost-orbit $\{u(t):t\in G\}$.

THEOREM 3.2. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty closed convex subset of E. Let G be right reversible and $\Im=\{S(t):t\in G\}$ be a semigroup of a.n.t. on C. Suppose that $F(\Im)\neq\emptyset$ and let $\{u(t):t\in G\}$ be an almost-orbit of $\Im$. If $\omega_w(u)\subseteq F(\Im)$, then the net $\{u(t):t\in G\}$ converges weakly to an element of $F(\Im)$.*

Proof. Be similar to Theorem 3 of [7].

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