Pusan Kyŏngnam Math. J. 8(1992), No 2, pp. 235–240

ON AN ANALYTIC CONTINUATION OF THE MULTIPLE HURWITZ ζ –FUNCTION

TAE-YOUNG SEO AND BO-MYOUNG OK

1. Multiple Hurwitz ζ-function and multiple Bernoulli polynomials

In [2], E.W. Barnes defines the r-ple Hurwitz ζ-function, for $\operatorname{Re} s > r$.

$$(1) \qquad \zeta_r(s, a|w_1, w_2, \cdots, w_r) = \sum_{m_1, m_2, \cdots, m_r = 0}^{\infty} \frac{1}{(a+\omega)^s},$$

where $\omega = m_1 w_1 + m_2 w_2 + \cdots + m_r w_r$ and also gives a contour integral representation

$$\zeta_r(s, a|w_1, w_2, \cdots, w_r) = \frac{i\Gamma(1-s)}{2\pi} \int_L \frac{e^{-az}(-z)^{s-1}}{\prod_{k=1}^{r}(1-e^{-w_k z})}\, dz,$$

where the conditions for a and $w_1, w_2, \cdots, w_r$ and the possible contour L is given by [2].

DEFINITION 1. In (1), we restrict these when $w_1 = w_2 = \cdots = w_n = 1$, that is to say, $a > 0$, $\operatorname{Re} s > n$

$$(2) \qquad \zeta_n(s, a) = \sum_{k_1, k_2, \cdots, k_n = 0}^{\infty} (a + k_1 + k_2 + \cdots + k_n)^{-s}$$

$\zeta_n(s, a)$ is called as the n-ple multiple Hurwitz ζ-function.

Received November 5, 1992. Revised

THEOREM 2. ([2]) *$\zeta_n(s,a)$ can be continued to a meromorphic function with poles $s = 1, 2, \cdots, n$, $a > 0$.*

Proof. For by the contour integral representation

$$\zeta_n(s,a) = \frac{i\Gamma(1-s)}{2\pi} \int_{\mathbf{c}} \frac{e^{-at}(-z)^{s-1}}{(1-e^{-z})^n} dz,$$

where the contour $\mathbf{C}$ is given as Fig.1, the integral is valid for $a > 0$ and all s, so $\zeta_n(s,a)$ has possible poles only at the poles of $\Gamma(1-s)$, i.e., $s = 1, 2, 3, \cdots$. But by the series definition $\zeta_n(s,a)$ is analytic for $\mathrm{Re}\, s > n$.

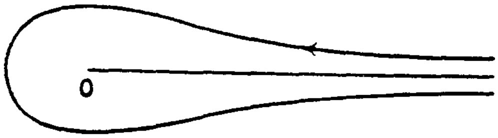

Fig.I

In particular, when $n = 1$, $\zeta_1(s,a) = \sum_{k_1=0}^{\infty}(a+k_1)^{-s} = \zeta(s,a)$. This is the well-known Hurwitz ζ-function.

DEFINITION 3. We define the k-th Bernoulli polynomials of order n, ${}_nB_k(a)$, whose first derivative ${}_nB_k^{(1)}(a)$ appears as the cofficient of z^k in the expansion

$$\begin{aligned}\frac{(-1)^n z e^{-az}}{(1-e^{-z})^n} &= \frac{(-1)^n A_n(a)}{z^{n-1}} + \frac{(-1)^{n-1}A_{n-1}(a)}{z^{n-2}} + \cdots \\ &+ \frac{A_2(a)}{z} - A_1(a) + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!}\, {}_nB_k^{(1)}(a) z^k \end{aligned} \tag{3}$$

which is valid in the annulus $\{z \mid 0 < |z| < 2\pi\}$.

Now, ${}_nB_k(a)$ is called the Multiple Bernoulli polynomial.

THEOREM 4.

$$A_s(a) = {}_nB_1^{(s+1)}(a) \quad \text{and} \quad \frac{{}_nB_{k+1}^{(2)}(a)}{k+1} = {}_nB_k^{(1)}(a)$$

for $k = 1, 2, \cdots$.

Proof. We differentiate (3) with regard to a ; we obtain

$$\frac{(-1)^n z e^{-az}}{(1-e^{-z})^n} = \frac{(-1)^n A'_{n-1}(a)}{z^{n-1}} + \frac{(-1)^{n-1} A'_{n-2}(a)}{z^{n-2}} + \cdots$$
$$+ \frac{(-1)A'_2(a)}{z^2} + \frac{A'_1(a)}{z} + \sum_{k=1}^{\infty} \frac{(-1)^k {}_nB_k^{(2)}(a)}{k!} z^{k-1}$$

Equatting now coefficient of like powers of z in (3) and we get $A_{n-q+1}(a) = A'_{n-q}(a)$, $q = 1, 2, \cdots, n-1$ and $A_1(a) = {}_nB_1^{(2)}(a)$. Hence $A_s(a) = {}_nB_1^{(s+1)}(a)$ and $\frac{{}_nB^{(2)}{}_{k+1}(a)}{k+1} = {}_nB'_k(a)$. Thus, the fundamental expansion (3) may be written

$$(4) \quad \frac{(-1)^n z e^{-az}}{(1-e^{-z})^n} = \sum_{s=1}^{n} \frac{(-1)^s {}_nB_1{}^{(s+1)}(a)}{z^{s-1}} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} {}_nB_k{}'(a)}{k!} z^k$$

2. An analytic continuation of the multiple Hurwitz ζ-function

THEOREM 5. *$\zeta_n(s,a)$ is a meromorphic function with simple poles at $s = 1, 2, \cdots, n$.*

Proof. From (2), for $\operatorname{Re} s > n$,

$$\zeta_n(s,a) = \sum_{k_1, k_2, \cdots, k_n = 0}^{\infty} (a + k_1 + k_2 + \cdots + k_n)^{-s}$$

We know that

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t}\, dt = (a + k_1 + \cdots + k_n)^s \int_0^\infty e^{-(a+k_1+\cdots+k_n)t} t^{s-1}\, dt$$

Then we have, for $\operatorname{Re} s > n$

$$\begin{aligned}\zeta_n(s,a)\Gamma(s) &= \int_0^\infty \frac{1}{(1-e^{-at})^n} e^{-at} t^{s-1}\, dt \\ &= \left(\int_0^1 + \int_1^\infty\right) t^{s-1} e^{-at} \frac{1}{(1-e^{-t})^n}\, dt\end{aligned}$$

Now, when $|s| \leq c$, where c is any positive number, we have

$$\begin{aligned}\int_1^\infty |\, t^{s-1} e^{-at} \frac{1}{(1-e^{-t})^n} \,|\, dt &= \int_1^\infty t^{c-1} e^{-at} \frac{1}{(1-e^{-t})^n}\, dt \\ &\leq \frac{1}{1-e^{-1}} \int_1^\infty t^{c-1} e^{-at}\, dt\end{aligned}$$

There, the second integral in (5) converges uniformly in every compact subset in the whole complex plane C and so represents an analytic function in C. On the ohter hand, the function $\frac{e^{t(n-a)}}{(e^t-1)^n}$ is analytic in a deleted neighborhood of zero and

$$\lim_{t\to 0} t^n \frac{e^{t(n-a)}}{(e^t-1)^n} = \lim_{t\to 0} \frac{te^{\frac{t(n-a)}{n}}}{e^t - 1} = 1 \neq 0,$$

but

$$\lim_{t\to 0} t^{n+1} \frac{e^{t(n-a)}}{(e^t-1)^n} = 0.$$

Thus $\frac{e^{t(n-a)}}{(e^t-1)^n}$ has a pole of order n at zero. Also, by (4), for $0 < |t| < 2\pi$

$$\begin{aligned}\frac{t^{s-1}e^{-at}}{(1-e^{-t})^n} &= {}_nB_1^{(n+1)}(a)t^{s-n-1} - {}_nB_1^{(n)}(a)t^{s-n} + \cdots \\ &+ (-1)^{n+2}\, {}_nB_1^{(3)}(a)t^{s-3} + (-1)^{n+1}\, {}_nB_1^{(2)}(a)t^{s-2} \\ &+ \sum_{k=1}^\infty \frac{(-1)^{n+k-1}\, {}_nB_k^{\prime}(a)}{k!} t^{k+s-2}.\end{aligned}$$

Using this expansion and term by term integration (justified by uniform convergence) the first integral in (5) can be written

$$\begin{aligned}\int_0^1 & t^{s-1}\frac{e^{-at}}{(1-e^{-t})^n}\,dt\\ &= \frac{{}_nB_1^{(n+1)}(a)}{s-n} - \frac{{}_nB_1^{(n)}(a)}{s-n+1} + \cdots + \frac{(-1)^{n+2}\,{}_nB_1^{(3)}(a)}{s-2}\\ &\quad + \frac{(-1)^{n+1}\,{}_nB_1^{(2)}(a)}{s-1} + \sum_{k=1}^{\infty}\frac{1}{k+s-1}\frac{(-1)^{n+k-1}\,{}_nB_k^{(1)}(a)}{k!}.\end{aligned}$$

Consequently, for $\operatorname{Re} s > n$, we can write

$$\begin{aligned}&\zeta_n(s,a)\\ &\quad = \frac{1}{\Gamma(s)}\left[\frac{{}_nB_1^{(n+1)}(a)}{s-n} - \frac{{}_nB_1^{(n)}(a)}{s-n+1} + \cdots + \frac{(-1)^{n+1}\,{}_nB_1^{(2)}(a)}{s-1}\right]\\ (6)\qquad &\quad + \frac{1}{\Gamma(s)}\left[\sum_{k=1}^{\infty}\frac{1}{k+s-1}\frac{(-1)^{n+k-1}\,{}_nB_k^{(1)}(a)}{k!}\right]\\ &\quad + \frac{1}{\Gamma(s)}\int_1^{\infty}\frac{t^{s-1}e^{-at}}{(1-e^{-t})^n}\,dt.\end{aligned}$$

As said before the thirdterm on the right in (5) is entire, and the series $\sum_{k=1}^{\infty}\frac{1}{k+s-1}\frac{(-1)^{n+k-1}\,{}_nB_k^{(1)}(a)}{k!}$ is meromorphic in the complex plane with simple poles at $-k$ if ${}_nB_k^{(1)}(a) \neq 0$, $k = 0, 1, 2, \cdots$. Since $\frac{1}{\Gamma(s)}$ is entire with zeros at $0, 1, 2, \cdots$, the right hand side of (6) is meromorphic on all of C with simple poles at $s = 1, 2, \cdots, n$.

COROLLARY 5. *The residue of $\zeta_n(s,a)$ at $s = r$ $(r = 1, 2, \cdots, n)$ is* $\frac{1}{(r-1)!}(-1)^{r+n}{}_nB_1^{(r+1)}(a)$.

Proof. From (6),

$$\begin{aligned}\lim_{s\to r}(s-r)\zeta_n(s,a) &= \frac{1}{\Gamma(r)}(-1)^{r+n}\,{}_nB_1^{(r+1)}(a)\\ &= \frac{1}{(r-1)!}(-1)^{r+n}\,{}_nB_1^{(r+1)}(a).\end{aligned}$$

References

1. J. Bak and D.J. Newman, *Complex Analysis*, Springer- Verlag, New York, 1982.
2. E.W. Barnes, *On the theory of the multiple gamma function*, Philosophical Transactions of the Royal Society(A) XIX (1904), 374–439.
3. J. Choi, *Determinants of Laplacians and multiple gamma functions*, Ph.D. Dissertation, Flodia State Univ., 1991.
4. M. Rao and H. Stetkær, *Complex Analysis*, World Scientific Publishing, 1991.
5. E.T. Whittaker and G.N. Watson, *A Course of Morden Analysis* (4th. Ed.), Cambridge Univ. Press, 1963.

Department of Mathematics
Pusan National University
Pusan 609–735, Korea