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FUZZY RELATIONS ON BCK/BCI—ALGEBRAS

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1. Introduction and preliminaries

The notion of a fuzzy subset and a fuzzy relation on a set was introduced by Zadeh ([6],[7]). Fuzzy relations on a group have been studied by Bhattacharya and Mukherjee ([1]). The concept of a fuzzy subalgebra of a BCK-algebra was introduced by Xi ([5]). In [3] the second author together with J. Meng solved the problem of classifying fuzzy subalgebras by their family of level subalgebras in BCK/BCI-algebras. In this paper we study fuzzy relations on BCK/BCI-algebras. We prove the following results. (i) If μ and σ are fuzzy subalgebras of a BCK/BCI-algebra X, then $\mu \times \sigma$ is a fuzzy subalgebra of $X \times X$. (ii) If $\mu \times \sigma$ is a fuzzy subalgebra of $X \times X$, then either μ or σ is a fuzzy subalgebra of X. (iii) If σ is a fuzzy subset of a BCK/BCI-algebra X and μ_σ is the strongest fuzzy relation on X that is a fuzzy relation on σ, then μ_σ is a fuzzy subalgebra if and only if σ is a fuzzy subalgebra. An example is given to show that if $\mu \times \sigma$ is a fuzzy subalgebra of $X \times X$, then μ and σ both need not be fuzzy subalgebras of X.

We recall some definitions and results.

DEFINITION 1.1. A *fuzzy subset* of any set S is a function $\mu : S \to [0,1]$.

DEFINITION 1.2. ([2]) Let μ be a fuzzy subset of a set S. For $t \in [0,1]$, the set

$$\mu_t := \{x \in S | \mu(x) \geq t\}$$

is called a *level subset* of μ.

DEFINITION 1.3. ([3],[5]) Let X be a BCK/BCI-algebra. A fuzzy subset μ of X is called a *fuzzy subalgebra* of X if for all $x, y \in X$,

$$\mu(x * y) \geq \min(\mu(x), \mu(y)).$$

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LEMMA 1.4. ([3]) *Let X be a BCK/BCI-algebra and let μ be a fuzzy subset of X such that μ_t is a subalgebra of X for all $t \in [0,1]$ $t \leq \mu(0)$. Then μ is a fuzzy subalgebra of X.*

DEFINITION 1.5. ([3]) Let X be a BCK/BCI-algebra and let μ be a fuzzy subalgebra of X. The subalgebras μ_t, $t \in [0,1]$ and $t \leq \mu(0)$, are called *level subalgebras* of μ.

DEFINITION 1.6. ([1]) Let S be any set. A *fuzzy relation* μ on S is a fuzzy subset of $S \times S$, that is, a map $\mu : S \times S \to [0,1]$.

DEFINITION 1.7. ([1]) If μ is a fuzzy relation on a set S and σ is a fuzzy subset of S, then μ is a *fuzzy relation on* σ if

$$\mu(x,y) \leq \min(\sigma(x), \sigma(y))$$

for all $x, y \in S$.

DEFINITION 1.8. ([1]) Let μ and σ be fuzzy subsets of a set S The *Cartesian product* of μ and σ is defined by

$$(\mu \times \sigma)(x,y) = \min(\mu(x), \sigma(y))$$

for all $x, y \in S$.

LEMMA 1.9. ([1]) *Let μ and σ be fuzzy subsets of a set S. Then*
(i) *$\mu \times \sigma$ is a fuzzy relation on S,*
(ii) *$(\mu \times \sigma)_t = \mu_t \times \sigma_t$ for all $t \in [0,1]$.*

DEFINITION 1.10. ([1]) If σ is a fuzzy subset of a set S, the *strongest fuzzy relation on* S that is a fuzzy relation on σ is μ_σ, given by

$$\mu_\sigma(x,y) = \min(\sigma(x), \sigma(y))$$

for all $x, y \in S$.

LEMMA 1.11. ([1]) *For a given fuzzy subset σ of a set S, let μ_σ be the strongest fuzzy relation on S. Then for $t \in [0,1]$, we have that*

$$(\mu_\sigma)_t = \sigma_t \times \sigma_t.$$

2. Fuzzy relations on BCK/BCI-algebras

LEMMA 2.1. ([5]) *If μ is any fuzzy subalgebra of a* BCK/BCI-*algebra X, then $\mu(0) \geq \mu(x)$ for all $x \in X$.*

PROPOSITION 2.2. *Let μ be a fuzzy subalgebra of a* BCK/BCI-*algebra X and let $x \in X$. If $\mu(x * y) = \mu(y)$ for every $y \in X$, then $\mu(x) = \mu(0)$.*

Proof. For a fixed element $x \in X$, suppose that $\mu(x * y) = \mu(y)$ for every $y \in X$. Choosing $y = 0$; then we have that $\mu(x) = \mu(x*0) = \mu(0)$.

PROPOSITION 2.3. *For a given fuzzy subset σ of a* BCK/BCI-- *algebra X, let μ_σ be the strongest fuzzy relation on X. If μ_σ is a fuzzy subalgebra of $X \times X$, then $\sigma(x) \leq \sigma(0)$ for all $x \in X$.*

Proof. From the fact that μ_σ is a fuzzy subalgebra of $X \times X$, it follows from Lemma 2.1 that for every $x \in X$,

$$\mu_\sigma(x, x) \leq \mu_\sigma(0, 0), \tag{*1}$$

where $(0, 0)$ is the zero element of $X \times X$. But (*1) means that

$$\min(\sigma(x), \sigma(x)) \leq \min(\sigma(0), \sigma(0)),$$

which implies that $\sigma(x) \leq \sigma(0)$.

The following proposition is an immediate consequence of Lemma 1.11, and we omit the proof.

PROPOSITION 2.4. *If σ is a fuzzy subalgebra of a* BCK/BCI-*algebra X, then the level subalgebras of μ_σ are given by $(\mu_\sigma)_t = \sigma_t \times \sigma_t$ for all $t \in [0, 1]$.*

THEOREM 2.5. *Let μ and σ be fuzzy subalgebras of a* BCK/BCI-*algebra X. Then $\mu \times \sigma$ is a fuzzy subalgebra of $X \times X$.*

Proof. For any $(x, y), (u, v) \in X \times X$, we have that

$$\begin{aligned}
&(\mu \times \sigma)((x, y) * (u, v)) \\
&= (\mu \times \sigma)(x * u, y * v) \\
&= \min(\mu(x * u), \sigma(y * v)) \\
&\geq \min(\min(\mu(x), \mu(u)), \min(\sigma(y), \sigma(v))) \\
&= \min(\min(\mu(x), \sigma(y)), \min(\mu(u), \sigma(v))) \\
&= \min((\mu \times \sigma)(x, y), (\mu \times \sigma)(u, v)).
\end{aligned}$$

This completes the proof.

THEOREM 2.6. *Let μ and σ be fuzzy subsets of a* BCK/BCI*-algebra X such that $\mu \times \sigma$ is a fuzzy subalgebra of $X \times X$. Then either μ or σ is a fuzzy subalgebra of X.*

Proof. Assume that μ and σ both are not fuzzy subalgebras of X. Then

$$\mu(x * y) < \min(\mu(x), \mu(y)) \quad \text{and} \quad \sigma(u * v) < \min(\sigma(u), \sigma(v))$$

for some $x, y, u, v \in X$. Now

$$\begin{aligned}
&(\mu \times \sigma)((x, u) * (y, v)) \\
&= (\mu \times \sigma)(x * y, u * v) \\
&= \min(\mu(x * y), \sigma(u * v)) \\
&< \min(\min(\mu(x), \mu(y)), \min(\sigma(u), \sigma(v))) \\
&= \min(\min(\mu(x), \sigma(u)), \min(\mu(y), \sigma(v))) \\
&= \min((\mu \times \sigma)(x, u), (\mu \times \sigma)(y, v)),
\end{aligned}$$

which is a contradiction. This completes the proof.

Now we give an example to show that if $\mu \times \sigma$ is a fuzzy subalgebra of $X \times X$, then μ and σ both need not be fuzzy subalgebras of X.

EXAMPLE. Let X be a nonzero BCK/BCI-algebra and let $t, s \in [0, 1]$ be such that $0 \leq s \leq t < 1$. Define fuzzy subsets $\mu, \sigma : X \to [0, 1]$ by $\mu(x) = s$ and

$$\sigma(x) = \begin{cases} t & \text{if } x = 0, \\ 1 & \text{if } x \neq 0, \end{cases}$$

for all $x \in X$, respectively. Then $(\mu \times \sigma)(x, y) = \min(\mu(x), \sigma(y)) = s$ for all $(x, y) \in X \times X$, that is, $\mu \times \sigma : X \times X \to [0, 1]$ is a constant function. Hence $\mu \times \sigma$ is a fuzzy subalgebra of $X \times X$. Now μ is a fuzzy subalgebra of X, but σ is not a fuzzy subalgebra of X since for $x \neq 0$ we have $\sigma(x * x) = \sigma(0) = t < 1 = \min(\sigma(x), \sigma(x))$.

THEOREM 2.7. *Let σ be a fuzzy subset of a* BCK/BCI*-algebra X. Then σ is a fuzzy subalgebra of X if and only if μ_σ is a fuzzy subalgebra of $X \times X$.*

Proof. ($\Longrightarrow$) Assume that σ is a fuzzy subalgebra of X. We claim that for any $(x_1, x_2), (y_1, y_2) \in X \times X$,

$$\mu_\sigma((x_1, x_2) * (y_1, y_2)) \geq \min(\mu_\sigma(x_1, x_2), \mu_\sigma(y_1, y_2)).$$

Since σ is a fuzzy subalgebra, we have that

$$\sigma(x_1 * y_1) \geq \min(\sigma(x_1), \sigma(y_1))$$

and

$$\sigma(x_2 * y_2) \geq \min(\sigma(x_2), \sigma(y_2)).$$

Hence

$$\begin{aligned}
&\mu_\sigma((x_1, x_2) * (y_1, y_2)) \\
&= \mu_\sigma(x_1 * y_1, x_2 * y_2) \\
&= \min(\sigma(x_1 * y_1), \sigma(x_2 * y_2)) \\
&\geq \min(\min(\sigma(x_1), \sigma(y_1)), \min(\sigma(x_2), \sigma(y_2))) \\
&= \min(\min(\sigma(x_1), \sigma(x_2)), \min(\sigma(y_1), \sigma(y_2))) \\
&= \min(\mu_\sigma(x_1, x_2), \mu_\sigma(y_1, y_2)),
\end{aligned}$$

and so the necessity is completed.

($\Longleftarrow$) Suppose that μ_σ is a fuzzy subalgebra of $X \times X$. Let $x_i, y_i \in X; i = 1, 2$. Then

$$\begin{aligned}
\mu_\sigma(x_1 * y_1, x_2 * y_2) &= \mu_\sigma((x_1, x_2) * (y_1, y_2)) \\
&\geq \min(\mu_\sigma(x_1, x_2), \mu_\sigma(y_1, y_2)).
\end{aligned}$$

This means that

$$\min(\sigma(x_1 * y_1), \sigma(x_2 * y_2)) \geq \min(\min(\sigma(x_1), \sigma(x_2)), \min(\sigma(y_1), \sigma(y_2))),$$

which implies that

$$\sigma(x_1 * y_1) \geq \min(\min(\sigma(x_1), \sigma(x_2)), \min(\sigma(y_1), \sigma(y_2))).$$

In particular, if we take $x_2 = 0 = y_2$, then by Proposition 2.3,

$$\begin{aligned}
\sigma(x_1 * y_1) &\geq \min(\min(\sigma(x_1), \sigma(0)), \min(\sigma(y_1), \sigma(0))) \\
&= \min(\sigma(x_1), \sigma(y_1)).
\end{aligned}$$

Hence σ is a fuzzy subalgebra of X.

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