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CODIMENSION 2 FIBRATORS IN MANIFOLD DECOMPOSITIONS

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1. Introduction

This paper presents that a proper map defined on an arbitrary manifold of codimension two can be recognized as an *approximate fibration*, by virtue of having all point preimages of a certain fixed homotopy type. Coram and Duvall initiated this type of investigation. After introducing the concept of an *approximate fibration* ([1]), they showed that a proper map $p : S^3 \mapsto S^2$ for which each $p^{-1}(b)$ is homotopy equivalent to S^1 is an *approximate fibration* over the complement of at most two points ([3]).

A proper map $p : M \mapsto B$ between locally compact ANRs is called an *approximate fibration* if it has the following homotopy property: Given an open cover ϵ of B ,an arbitrary space X and two maps $g : X \mapsto M$ and $F : X \times I \mapsto B$ such that $p \circ g = F_0$, there exists a map $G : X \times I \mapsto M$ such that $G_0 = g$ and $p \circ G$ is ϵ - close F.

The continuity set of $p : M \mapsto B$, denoted by C, consists of all points of $x \in B$ such that,under any retraction $R : p^{-1}(U_x) \mapsto p^{-1}(x)$ defined over a neighborhood $U_x \subset B$ of x as x has another neighborhood $V_x \subset U_x$ such that for all $b \in V_x$, $R \mid p^{-1}(b) : p^{-1}(b) \mapsto p^{-1}(x)$ is a degree one map. Coram and Duvall have shown C to be a dense, open subset of B ([3]).

Manifolds are taken to be finite dimensional, connected, boundaryless and orientable. A closed manifold N is a *codimension 2 fibrator* if, whenever $p : M \mapsto B$ is a proper map from an (n+2)–manifold M to an 2–manifold B such that each $p^{-1}(b)$ is shape equivalent to N, p is an *approximate fibration*.

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Homology is computed with integer coefficients unless the coefficient module is mentioned.

R.J. Daverman has shown that all closed 2-manifold with non-zero Euler characteristic are *codimension 2 fibrators* ([4]), and Y.H. Im proved that any finite product of closed, orientable surfaces of genus at least 2 is also a *codimension 2 fibrator* ([8]). Recently R.J. Daverman investigated which 3-manifolds can be *codimension 2 fibrators* by their geometric structure, except 3-manifolds with spherical geometry ([5]).

In this paper, we show that any closed manifold N^n covered by n-sphere is a *codimenstion 2 fibrator*, and as a result,a closed 3-manifold with spherical geometry is a *codimension 2 fibrator*.

2. Preliminaries

The following result is the key fact to investigate *codimension 2 fibrators*, which is found in [6].

LEMMA 2.1. *If G is a usc decomposition of an orientable $(n+2)$-manifold M into closed, orientable n-manifolds, then the decomposition space B is a 2-manifold and $D = B - C$ is locally finite in B, where C represents the continuity set of $p : M \mapsto B$.*

As a result, we can localize the problem to that of an open disk of B, provided that $p : M \mapsto B$ is an approximate fibration over the continuity set C, where $p : M \mapsto B$ is an *approximate fibration* over $B - b_0$ for some $b_0 \in B$.

LEMMA 2.2. ([4]) *Suppose N^n is a closed, orientable manifold with finite fundamental proup and G is a usc decomposition of M^{n+2} into copies of N^n. Then $p : M^{n+2} \mapsto B(= M/G)$ is an approximate fibration over its continuity set.*

Now, we give an algebraic criterion which is necessary to determine whether a certain closed manifold can be a codimension 2 fibrator or not. Let G be a usc decomposition of an arbitrary $(n+2)$-manifold M into copies of a closed manifold N^n, a retraction $R : U \mapsto g_0$ is defined on some open set $U \supset g_0$ for each $g_0 \in G$, and $p : M \mapsto B$ be a decomposition map.

LEMMA 2.3. *Assume that for all $g \in G$ sufficiently close to a fixed $g_0 = p^{-1}(b_0)$, and a given neiborhood retraction $R : U \mapsto g_0$, the restriction $R| : g \mapsto g_0$ induces $H_1 - isomorphism$ $(R|)_* : H_1(g) \mapsto H_1(g_0)$. Then $R| : g \mapsto g_0$ is a degree one map.*

Proof. Consider the following diagram

$$\begin{array}{ccccccc} Z \cong H_2(M, M-g_0) & \xrightarrow{\partial} & H_1(M-g_0) & \longrightarrow & H_1(M) & \longrightarrow & 0 \\ {\scriptstyle p_*}\downarrow & & {\scriptstyle (p_1)_*}\downarrow & & & & \\ Z \cong H_2(B, B-b_0) & \xrightarrow{\cong} & H_1(B-b_0) & & & & \end{array}$$

From the Alexander duality, we have

$$H_2(M, M-g_0) \cong H^n(g_0) \cong Z$$

and

$$H_2(B, B-b_0) \cong H^0(b_0) \cong Z.$$

Since $(R|)_* : H_1(g) \mapsto H_1(g_0)$ is an isomorphism, it easily follows that

$$H_1(M-g_0) \cong \lambda_* H_1(g) \oplus im\partial,$$

where $\lambda_* : g \mapsto M - g_0$. From the diagram chasing, we see that

$$p_* : H_2(M, M-g_0) \mapsto H_2(B, B-b_0)$$

is an isomorphism,by using $\lambda_*(H_1(g)) \subset ker(p_1)_*$. By a similar way, we obtain an isomorphism $p_* : H_2(M, M-g) \mapsto H_2(B, B-c)$ for $c = p(g)$. Then the following commutative diagram holds, where V is a connected open neiborhood of b_0 ($\cong R^2$) having compact closure and $c \in V$.

$$\begin{array}{ccccc} H^n(g_0) \cong H_2(M, M-g_0) & \xrightarrow{\cong} & H_2(B, B-b_0) & \cong & H^0(b_0) \\ \uparrow & & \uparrow & & \uparrow{\scriptstyle \cong} \\ H_2(M, M-cl(p^{-1}U)) & \xrightarrow{\cong} & H_2(B, B-cl(U)) & \cong & H^0(clU) \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \cong} \\ H^n(p^{-1}(c)) \cong H_2(M, M-p^{-1}c) & \xrightarrow{\cong} & H_2(B, B-c) & \cong & H^0(c) \end{array}$$

This implies that $(R|)^*$ on the cohomology is constantly 1 near b_0, and implies the same on the homology by the universal coefficient theorem. Therefore the conclusion follows.

3. Main result

The main theorem in this section shows that all closed manifold covered by n-spheres are *codimension* 2 *fibrators*.

THEOREM 3.1. *Suppose N^n is a closed manifold covered by n– sphere. Then N^n is a codimension 2 fibrator.*

Proof. Lemma 2.1 implies that the decomposition space B is a 2-manifold and the discontinuity set $D = B - C$ of $p : M \mapsto B$ is locally finite. By Lemma.2.2, one can localize the problem in which B is an open disk and p is an approximate fibration over the complement of one point b_0.

Set $g_0 = p^{-1}(b_0)$. By the property of approximate fibration $p|M - g_0 : M - g_o \mapsto B - b_0$, we have a strong deformation retraction $R : M \mapsto g_0$. Now define $r = R|g : g \mapsto g_0$, where $g = p^{-1}(b)$ and $b \in B - b_0$.

Then we have the following homotopy exact sequence of an approximate fibratoion ([1]).

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(g) & \longrightarrow & \pi_1(M - g_0) & \longrightarrow & \pi_1(B - b_0) & \longrightarrow & 1 \\ & & & & & & \| & & \\ & & & & & & Z & & \end{array}$$

Choose a subgroup K of $\pi_1(M - g_0)$ such that $K \cong Z$ and the index $[\pi_1(M - g_0) : K]$ is equal to the order of $\pi_1(g)$.

For an approximate fibration $p|M - g_o : M - g_o \mapsto B - b_0$, there is a corresponding Hurewicz fibration $p' : M' \mapsto B - b_0$ ([7], Theorem 1.2). Here $M - g_0$ and M' are homotopy equivalent. We consider the restriction $p''(= p'|M_0) : M_0 \mapsto S^1$, where $M_0 = p'^{-1}(S^1)$ Then p'' is a Hurewicz fibration. Without loss of generality we can assume $p'' : M_0 \mapsto S^1$ is a fiber bundle with fiber N^n.

We take a covering $q : M_0^* \mapsto M$ corresponding to the group K. Then $q \circ p'' : M_0^* \mapsto S^1$ is a fiber bundle. It can be easily checked that fiber of $q \circ p''$ is S^n because N^n is covered by S^n. From [9, Corollary

18.6] $q \circ p'' : M_0^* \mapsto S^1$ is equivalent to the trivial bundle. Thus the following commutative diasgram holds.

$$\begin{array}{ccc} M_0^* & \overset{h}{\longleftarrow} & S^n \times S^1 \\ {\scriptstyle q\circ p''}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ S^1 & = \!= \!= & S^1 \end{array},$$

where h is a fiber preserving homeomorphism and π is the projection onto the second factor.

We prove that the covering $q : M_0^* \mapsto M_0$ is a regular covering. For any covering transformation $\varphi : S^n \times S^1 \mapsto S^n \times S^1$, and any $x \in M_0$, φ operates transitively on $(h \circ q)^{-1}(x)$. This implies that $q \circ h$ is a regular covering, and hence q is a regular covering. Therefore, K is a normal subgroup. As a result,

$$\pi_1(M_0) = \pi_1(M - g_0) = \pi_1(N) \times \pi_1(B - b_0).$$

Then we can see easily that $r = R|g : g \mapsto g_0$ induces a π_1-isomorphism. By Lemma 2.3, r is a degree one map. The conclusion follows from Lemma 2.2.

As a consequence, we have the following immediate corollaries.

COROLLARY 3.2. *Every Lens space is a codimension 2 fibrator.*

COROLLARY 3.3. *Every closed 3-manifold with spherical geometry is a codimension 2 fibrator.*

References

1. D. Coram and P. Duvall, *Approximate fibration*, Rocky mountain J. Math. **7** (1977), 275–288
2. D. Coram and P. Duvall, *Approximate fibrations and a movability condition for maps*, Pacific J. Math. **72** (1977), 41–56
3. D. Coram and P. Duvall, *Mappings from S^3 to S^2 whose point inverses have the shape of a circle*, Topology Appl. **10** (1979), 239–246
4. R.J. Daverman, *Submanifold decompositions that induce approximate fibrations*, Topology Appl. **33** (1989), 173–184
5. R.J. Daverman, 3*–manifolds with geometric structures and approximate fibrations* (to appear).

6. R.J. Daverman and J.J. Walsh, *Decompositions into codimension two manifolds*, Trans Amer. Math. Soc. **288** (1985), 273–291.
7. C.B. Hughes, L.R. Taylor and E.B. Williams, *Bundle theories for topological manifolds*, Trans. Amer. Math. Soc. **319** (1990), 1–65
8. Y.H. Im, *Decompositions into codimension two submanifolds that induce approximate fibrations* (to appear).
9. N. Steenrod, *The topology of fibre bundle*, Princeton Univ. Press, Princeton N.J., 1974.

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