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A CONSTRUCTION OF PIECEWISE SOLENOIDAL VECTOR FIELDS

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1. Introduction

Let Ω be a bounded domain in R^N with $N = 2$ or $N = 3$, and whose boundary $\partial\Omega$ is smooth. We consider the boundary value problem of the stationary Stokes equations: Seek a vector valued function $\vec{u} = (u_1, u_2, \ldots, u_N)$ (the velocity) and a scalar function p (the pressure) satisfying

$$(1) \qquad \begin{cases} -\nu\Delta\vec{u} + \text{grad } p = \vec{f} & \text{in } \Omega, \\ \text{div } \vec{u} = 0 & \text{in } \Omega, \\ \vec{u} = 0 & \text{on } \partial\Omega; \end{cases}$$

where $\nu > 0$ is the kinematic viscosity and $\vec{f} = (f_1, f_2, \ldots, f_N)$ is a given vector valued function.

For results concerning existence,uniqueness and regularity of (weak) solutions of (1), we refer to [6].

In constructing Galerkin discretization for the Stokes problem, one encounters a major difficulty in incorporating the incompressibility condition into the finite element space (cf.[5]). Various techniques have been developed to avoid this difficulty. One such method consists in using a Lagrange multiplier technique (cf.[3,4]). In [1], they introduced the finite dimensional approximating spaces V_k^N consisting of piecewise polynomial functions of degree $\leq k$, $k \geq 1$ for the velocity $\vec{u}$ that are piecewise solenoidal i.e. the constituent functions satisfy the incompressibility condition (strongly) on each triangle in the subdivision of the domain Ω. And they proved that these functions possess optimal approximating properties on the domains with curved boundaries.

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In this paper, we compute the dimension of V_k^N and construct the basis of V_k^N to provide an opportunity to use computer codes for real computations. In this paper, we will denote the set of polynomials of degree $\leq k$ by P_k i.e.

$$P_k = \left\{ \sum \gamma_{\alpha_1\alpha_2\ldots\alpha_N} x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_N^{\alpha_N} \mid \sum_{l=1}^{N} \alpha_l \leq k,\ \gamma_{\alpha_1\alpha_2\ldots\alpha_N} \in R \right\}$$

We know that (see [2])

$$\dim P_k = \binom{N+k}{k}.$$

2. The dimension of V_k^N

THEOREM 1. $V_k^N = \{p \in (P_k)^N \mid div\ p = 0\}$. *Then*

$$dim\ V_k^N = N\binom{N+k}{k} - \binom{N+k-1}{k-1}.$$

Proof. Let $A_k = \left\{ x_1^{r_1} x_2^{r_2} \ldots x_N^{r_N} \mid 0 \leq r_i \leq k,\ \sum_{i=1}^{N} r_i = k \right\}$. Then P_k is the collection of all the possible finite linear combination of elements in $A_0 \cup \cdots \cup A_k$, so we can say $P_k = \text{Span}\{A_0 \cup A_1 \cup \cdots \cup A_k\}$. Obviously,

$$A_{k+1} = \bigcup_{i=1}^{N} \{x_i a_k | a_k \in A_k\}.$$

First we'll show that for each $1 \leq i \leq N$,

$$\text{Span}(A_k) = \left\{ \frac{\partial}{\partial x_i} a_{k+1} | a_{k+1} \in \text{Span}(A_{k+1}) \right\}.$$

Choose $x_1^{r_1} x_2^{r_2} \ldots x_N^{r_N} \in A_k$. If $r_i = 0$, then $x_i(x_1^{r_1} x_2^{r_2} \ldots x_N^{r_N}) \in A_{k+1}$ and

$$\frac{\partial}{\partial x_i} [x_i(x_1^{r_1} x_2^{r_2} \ldots x_N^{r_N})] = x_1^{r_1} x_2^{r_2} \ldots x_N^{r_N}.$$

If $r_i \neq 0$, then $(x_1^{r_1} x_2^{r_2} \dots x_i^{r_i+1} \dots x_N^{r_N})/(r_i+1) \in \mathrm{Span}(A_{k+1})$, and

$$\frac{\partial}{\partial x_i}[(x_1^{r_1} x_2^{r_2} \dots x_i^{r_i+1} \dots x_N^{r_N})/(r_i+1)] = x_1^{r_1} x_2^{r_2} \dots x_i^{r_i} \dots x_N^{r_N}.$$

Therefore $A_k \subset \left\{ \frac{\partial}{\partial x_i} a_{k+1} | a_{k+1} \in \mathrm{Span}(A_{k+1}) \right\}$, which implies

$$\mathrm{Span}(A_k) \subset \left\{ \frac{\partial}{\partial x_i} a_{k+1} | a_{k+1} \in \mathrm{Span}(A_{k+1}) \right\}.$$

To prove the reverse inclusion, choose $x_1^{r_1} x_2^{r_2} \dots x_N^{r_N} \in A_{k+1}$, then there exist i_0 and $a_k \in A_k$ such that

$$x_{i_0} a_k = x_{i_0}(x_1^{s_1} x_2^{s_2} \dots x_N^{s_N}), \text{ where } 0 \leq s_i \leq k \text{ and } \sum_{i=1}^{N} = k.$$

If $i = i_0$,

$$\begin{aligned} \frac{\partial}{\partial x_i}(x_{i_0} a_k) &= a_k + (\frac{\partial}{\partial x_i} a_k) x_{i_0} \\ &= a_k + s_i a_k \\ &= (1 + s_i) a_k \end{aligned}$$

which is in $\mathrm{Span}(A_k)$.
If i$\neq i_0$,

$$\begin{aligned} \frac{\partial}{\partial x_i}(x_{i_0} a_k) &= x_{i_0} \frac{\partial}{\partial x}(a_k) \\ &= s_i x_{i_0} x_1^{s_1} \dots x_i^{s_i - 1} \dots x_N^{s_N} \end{aligned}$$

which is in $\mathrm{Span}(A_k)$. Therefore we showed,

$$\left\{ \frac{\partial}{\partial x_i} a_{k+1} | a_{k+1} \in \mathrm{Span}(A_{k+1}) \right\} \subset \mathrm{Span}(A_k).$$

Next, we will proved that div is a linear mapping from $(P_k)^N$ onto P_{k-1}, which implies the result

$$\begin{aligned} \dim V_k^N &= \dim(\ker(\mathrm{div})) \\ &= \dim (P_K)^N - \dim P_{k-1} \\ &= N\binom{N+k}{k} - \binom{N+k-1}{k-1}. \end{aligned}$$

First we take div to the polynomial in $(P_k)^N$.

$$\begin{aligned}
&\{\ \mathrm{div}(p_1, p_2, \dots, p_N) \mid p_i \in P_k \text{ for all } 1 \le i \le N\} \\
&= \left\{ \sum_{i=1}^{N} \frac{\partial p_i}{\partial x_i} \mid p_i \in P_k, 1 \le i \le N \right\} \\
&= \left\{ \sum_{i=1}^{N} \frac{\partial p_i}{\partial x_i} \mid p_i \in \mathrm{Span}(A_0 \cup A_1 \cup \dots \cup A_k),\ 1 \le i \le N \right\}.
\end{aligned}$$

Now choose $p_i \in A_{k_i}$ for some $1 \le k_i \le k$. Then

$$\begin{aligned}
\frac{\partial p_i}{\partial x_i} &\in \mathrm{Span}(A_0 \cup A_1 \cup \dots \cup A_{k^*-1}) \\
&\subset P_{k-1}
\end{aligned}$$

for $1 \le i \le N$, where k^*=max$\{k_1, k_2, \dots, k_N\} \le$ k. Let's show the reverse inclusion,

$$\begin{aligned}
P_{k-1} &= \mathrm{Span}(A_0 \cup A_1 \cup \dots \cup A_{k-1}) \\
&= \mathrm{Span} A_0 \oplus \mathrm{Span} A_1 \oplus \dots \oplus \mathrm{Span} A_{k-1} \\
&= \left\{ \frac{\partial a_1}{\partial x_1} | a_1 \in \mathrm{Span}(A_1) \right\} \\
&\oplus \left\{ \frac{\partial a_2}{\partial x_1} | a_2 \in \mathrm{Span}(A_2) \right\} \\
&\vdots \\
&\oplus \left\{ \frac{\partial a_k}{\partial x_1} | a_k \in \mathrm{Span}(A_k) \right\} \\
&= \left\{ \frac{\partial p}{\partial x_1} | p \in P_k \right\} \\
&= \left\{ \sum_{i=1}^{N} \frac{\partial}{\partial x_i} p_i | p_1 \in P_k \text{ and } p_i = 0, \text{ if } i \ne 1 \right\} \\
&\subset \{\ \mathrm{div}(p_1, p_2, \dots, p_N) \mid p_i \in P_k\}.
\end{aligned}$$

Therefore we proved div is a linear mapping from $(P_k)^N$ onto P_{k-1}.

3. The construction of a basis of V_k^N

Case I : N = 2.

(i) k = 1

Theorem 1 gives dim $V_1^2 = 5$. The basis is the following:

$$\begin{pmatrix}1\\0\end{pmatrix} \begin{pmatrix}0\\1\end{pmatrix} \begin{pmatrix}0\\x\end{pmatrix} \begin{pmatrix}y\\0\end{pmatrix} \begin{pmatrix}x\\-y\end{pmatrix}.$$

(ii) k = 2

The basis is the following:

The vector polynomials for k = 1, and

$$\begin{pmatrix}x^2\\-2xy\end{pmatrix} \begin{pmatrix}-2xy\\y^2\end{pmatrix} \begin{pmatrix}y^2\\0\end{pmatrix} \begin{pmatrix}0\\x^2\end{pmatrix}.$$

(iii) k = n

The basis is the set of vector polynomials for k = n - 1, and

$$\begin{pmatrix}0\\x^n\end{pmatrix} \begin{pmatrix}0\\y^n\end{pmatrix} \begin{pmatrix}x^n\\-nx^{n-1}y\end{pmatrix} \begin{pmatrix}2x^{n-1}y\\-(n-1)x^{n-2}y^2\end{pmatrix}$$

$$\begin{pmatrix}3x^{n-2}y^2\\-(n-2)x^{n-3}y^3\end{pmatrix} \cdots \begin{pmatrix}-nxy^{n-1}\\y^n\end{pmatrix}.$$

Case II : N = 3.

(i) k = 1

Theorem 1 gives dim $V_1^3 = 11$.

The basis is the following:

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\end{pmatrix} \begin{pmatrix}0\\0\\1\end{pmatrix} \begin{pmatrix}0\\x\\0\end{pmatrix} \begin{pmatrix}0\\0\\x\end{pmatrix} \begin{pmatrix}y\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}0\\0\\y\end{pmatrix} \begin{pmatrix}z\\0\\0\end{pmatrix} \begin{pmatrix}0\\z\\0\end{pmatrix} \begin{pmatrix}x\\0\\-z\end{pmatrix} \begin{pmatrix}0\\y\\-z\end{pmatrix}.$$

(ii) k = 2

The basis is the set of the vector polynomials for k = 1 and

$$\begin{pmatrix}0\\x^2\\0\end{pmatrix} \begin{pmatrix}0\\0\\x^2\end{pmatrix} \begin{pmatrix}y^2\\0\\0\end{pmatrix} \begin{pmatrix}0\\0\\y^2\end{pmatrix} \begin{pmatrix}z^2\\0\\0\end{pmatrix} \begin{pmatrix}0\\z^2\\0\end{pmatrix}$$

$$\begin{pmatrix} x^2 \\ 0 \\ -2xz \end{pmatrix} \begin{pmatrix} -2xz \\ 0 \\ z^2 \end{pmatrix} \begin{pmatrix} 0 \\ y^2 \\ -2yz \end{pmatrix} \begin{pmatrix} 0 \\ -2yz \\ z^2 \end{pmatrix} \begin{pmatrix} xy \\ 0 \\ -yz \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ xy \\ -xz \end{pmatrix} \begin{pmatrix} yz \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ xz \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ xy \end{pmatrix}.$$

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