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A YANG–MILLS CONNECTION ON (S^3, can)

JOON-SIK PARK

1. Introduction and statement of a result

To find *Yang-Mills connections* in some principal fibre bundle is important. In this paper, we find a *Yang-Mills connection* in the orthonomal frame bundle on (S^3, can). Since (S^3, can) and $(SU(2), (\ ,\ ))$ are homothetic, $(\ ,\ )$ being an arbitrary biinvariant metric on $SU(2)$, we treat $(SU(2), (\ ,\ ))$ in place of the base manifold (S^3, can).

We prepare some notations. In this paper, we put $M := SU(2)$ and $G := O(3)$. Let $P(M, G)$ be the orthonormal frame bundle over $(M, (\ ,\ )_o)$. Here $(\ ,\ )_o$ is the biinvariant riemannian metric induced from $(-1)\cdot$ (Killing form of $\mathfrak{m}$). Here $\mathfrak{m}$ is the Lie algebra of M. We put

$$(1.1)\qquad X_1 := c\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X_2 := c\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$X_3 := c\begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \qquad \left(c = \frac{\sqrt{-1}}{\sqrt{8}}\right).$$

Then $\{X_1, X_2, X_3\}$ is an orthonormal basis of $\mathfrak{m}$ with respect to $(\ ,\ )_o$. The connection function α ([4], p.43) on $\mathfrak{m}\times\mathfrak{m}$ which is corresponding to the biinvariant riemannian connection of $(M, (\ ,\ )_o)$ is given as follows ([4], p.52):

$$(1.2)\qquad \alpha(X, Y) = \frac{1}{2}[X, Y], \qquad (X, Y \in \mathfrak{m}).$$

$\alpha(X_i, X_j)$ is uniquely expressed as

$$(1.3)\qquad \alpha(X_i, X_j) = \frac{1}{2}\sum_{k=1}^{3} c_{ij}{}^{k} X_k, \qquad (i, j = 1, 2, 3),$$

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where ${c_{ij}}^k$ is the structure constants of $\mathfrak{m}$ with respect to the orthonormal basis $\{X_1, X_2, X_3\}$. Let $\{\theta^1, \theta^2, \theta^3\}$ be the dual 1-forms to the basis $\{X_1, X_2, X_3\}$. Then, the connection form ω and the curvature form Ω with respect to frames are defined as follows:

$$\omega^i_j = \frac{1}{2}\sum_{k=1}^{3} {c_{kj}}^i \theta^k, \tag{1.4}$$

$$\begin{aligned} \Omega^i_j = \sum_{k<l} \theta^i(\alpha(X_k, \alpha(X_l, X_j)) - \alpha(X_l, \alpha(X_k, X_j)) \\ - \alpha([X_k, X_l], X_j))\theta^k \wedge \theta^l. \end{aligned} \tag{1.5}$$

We denote by $\mathcal{A}_P$ the totality of connections in the above given orthonormal frame bundle $P(M, G)$ which is a principal fibre bundle. We also denote by $\mathfrak{g}$ the Lie algebra of the structure group G of $P(M, G)$. The *Yang-Mills functional* E on $\mathcal{A}_P$ is defined by

$$E(A) = \frac{1}{2}\int_M \|F(A)\|^2$$

for each $A \in \mathcal{A}_P$, where $F(A)$ is the curvature form of a given connection A. In fact, the connection form ω of (1.4) with respect to frames belongs to $\mathcal{A}_P$, and $F(\omega)$ is equal to Ω of (1.5). We denote $P(M, G) \times_{Ad} \mathfrak{g}$ by $\mathfrak{g}_P$. Let $\Omega^r(\mathfrak{g}_P), 0 \le r \le 3$, be the space of $\mathfrak{g}_P$-valued r forms on M. The covariant exterior differentiation $d_A : \Omega^k(\mathfrak{g}_P) \to \Omega^{k+1}(\mathfrak{g}_P)$ for $A \in \mathcal{A}_P$ is defined by

$$d_A(\phi) = d\phi + [A \wedge \phi], \qquad (\phi \in \Omega^k(\mathfrak{g}_P)). \tag{1.6}$$

We denote also by δ_A the formal adjoint operator of d_A. It is well known that $\beta = A - A'$ belongs to $\Omega^1(\mathfrak{g}_P)$ for each A, $A' \in \mathcal{A}_P$, and a connection $A \in \mathcal{A}_P$ is a *Yang-Mills connection* (a critical point of the *Yang-Mills functional* E) if and only if

$$\delta_A F(A) = 0. \tag{1.7}$$

Now we state our main theorem.

THEOREM. *The connection (1.4) with respect to frames in the orthonormal frame bundle over $(SU(2),(\ ,\ )_o)$ is a Yang-Mills connection, i.e., the connection with respect to frames in the orthonormal frame bundle over (S^3, can) is a Yang-Mills connection.*

2. Proof of Theorem

We put $H_1 := 2\sqrt{2}X_1$, $U_1 := 2\sqrt{2}X_2$ and $V_1 := 2\sqrt{2}X_3$. Then, We have

$$(2.1)\qquad [H_1, U_1] = 2V_1, \quad [U_1, V_1] = 2H_1, \text{ and } [V_1, H_1] = 2U_1.$$

From (1.2), (1.3) and (2.1), we obtain

$$(2.2)\qquad \begin{aligned} &{c_{12}}^1 = {c_{13}}^1 = {c_{12}}^2 = {c_{23}}^2 = {c_{13}}^3 = {c_{23}}^3 = 0, \\ &{c_{23}}^1 = {c_{31}}^2 = {c_{12}}^3 = (1\sqrt{2}). \end{aligned}$$

Using (1.4), (1.5) and (2.2), we get

$$(2.3)\qquad (\omega^i_j) = \left(\frac{\sqrt{2}}{4}\right)\begin{pmatrix} 0 & -\theta^3 & \theta^2 \\ \theta^3 & 0 & -\theta^1 \\ -\theta^2 & \theta^1 & 0 \end{pmatrix}$$

$$(2.4)\qquad (\Omega^i_j) = \left(\frac{1}{8}\right)\begin{pmatrix} 0 & \theta^1\wedge\theta^2 & \theta^1\wedge\theta^3 \\ -\theta^1\wedge\theta^2 & 0 & \theta^2\wedge\theta^3 \\ -\theta^1\wedge\theta^3 & -\theta^2\wedge\theta^3 & 0 \end{pmatrix}$$

We denote $\big(\nabla_{X_j}F(\omega)\big)(X_i, X_k), \omega(X_j)$ and $F(\omega)(X_j, X_i)$ by $\nabla_j F_{ik}, A_j$ and F_{ji} respectively. Thus, the connection ω in (1.4) is a *Yang-Mills connection* if and only if

$$(2.5)\qquad (\delta_\omega F(\omega))(X_i) = \sum_{j=1}^{3}(\nabla_j F_{ji} + [A_j, F_{ji}]) = 0, \quad (1 \le i \le 3).$$

From (1.3), (2.2) and (2.4), we get

$$(2.6)\qquad \begin{cases} \nabla_1 F_{21} = \nabla_3 F_{23} = \left(\dfrac{\sqrt{2}}{4}\right)F_{13}, \\ \nabla_1 F_{13} = \nabla_2 F_{23} = \left(\dfrac{\sqrt{2}}{4}\right)F_{12}, \\ \nabla_2 F_{12} = \nabla_3 F_{13} = \left(\dfrac{\sqrt{2}}{4}\right)F_{32}, \\ \nabla_1 F_{23} = \nabla_2 F_{31} = \nabla_3 F_{12} = 0. \end{cases}$$

Moreover, we obtain from (2.3) and (2.4),

$$(2.7) \qquad \begin{cases} [A_1, F_{12}] = [A_3, F_{32}] = (\frac{\sqrt{2}}{32})\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\ [A_1, F_{13}] = [A_2, F_{23}] = (\frac{\sqrt{2}}{32})\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ [A_2, F_{12}] = [A_3, F_{13}] = (\frac{\sqrt{2}}{32})\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \\ [A_1, F_{23}] = [A_2, F_{13}] = [A_3, F_{12}] = O_3 \end{cases}$$

where O_3 denotes the zero matrix of order 3. We have from (2.6) and (2.7)

$$(2.8) \qquad \sum_{i=1}^{3}(\nabla_i F_{ij} + [A_i, F_{ij}]) = 0, \qquad (j = 1, 2, 3).$$

Hence, the connection ω with respect to frames in the orthonormal frame bundle over $(SU(2), (\ ,\ )_o)$ is a *Yang-Mills connection.*

REMARK. This theorem was previously known, and indeed J.P. Bourguignon and B. Lawson ([1], Theorem C on p.191) that this *Yang-Mills fields* on S^3 is one of only two with small norm. But, the method proving this theorem in this paper is different and algebraic.

References

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Department of Mathematics
Pusan University of Foreingn Studies
Pusan 608–060, Korea