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# ON COMMON FIXED POINTS OF ASYMPTOTICALLY REGULAR MAPPINGS

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## 1. Introduction

In 1976, G. Jungck ([6]) initially proved a common fixed point theorem for commuting mappings which generalizes the well-known Banach's fixed point theorem. This result has been generalized, extended and improved in various ways by many authors ([2], [7]–[9], [11]–[12] and [14]–[18]).

On the other hand, S. Sessa ([16]) introduced a generalization of commutativity, which is called weak commutativity, and proved some common fixed point theorems for weakly commuting mappings which generalize the results of K.M. Das and K.V. Naik ([2]). Recently, G. Jungck ([7]) introduced the concept of more generalized commutativity, so called compatibility, which is more general than that of weak commutativity. The utility of compatibility in the context of fixed point theory was initially demonstrated in extending a theorem of S. Park and J.S. Bae ([15]). By employing compatible mappings in stead of commuting mappings and using four mappings as opposed to three, G. Jungck ([8]) extended the results of M.S. Khan and M. Imdad ([12]), S.L. Singh and S.P. Singh ([18]) and, recently, also obtained an interesting result concerning with his concept in his consecutive paper ([9]). Quite recently, H. Kaneko and S. Sessa ([11]) also generalized the result of K.M. Das and K.V. Naik ([2]) by using the concept of compatibility.

In this paper, we prove some common fixed point theorems of asymptotically regular mappings by using compatible mappings. Our results generalize and improve the results of D.E. Anderson, K.L. Singh and J.H.M. Whitfield ([1]), M.D. Guay and K.L. Singh ([4]), G. Jungck ([8]), H. Kaneko and S. Sessa ([11]), R.N. Mukherjee ([14]) and others.

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## 2. Preliminaries

For some definitions, terminologies and notations in this paper, we refer to [1], [7], [16] and [17].

DEFINITION 2.1. Let $A$ and $B$ be mappings from a metric space $(X,d)$ into itself. Then the pair $(A,B)$ is said to be ***asymptotically regular*** at $x_0$ in $X$ if $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, where $\{x_n\}$ is a sequence in $X$ defined by $x_1 = Ax_0$, $x_2 = Bx_1, \cdots$, $x_{2n+1} = Ax_{2n}$, $x_{2n+2} = Bx_{2n+1}, \cdots$.

DEFINITION 2.2. Let $A$ and $B$ be mappings from a metric space $(X,d)$ into itself. Then $A$ and $B$ are said to be ***weakly commuting mappings*** on $X$ if $d(ABx, BAx) \le d(Ax, Bx)$ for all $x$ in $X$.

Clearly, any commuting mappings are weakly commuting, but the converse is not necessarily true as in the following example:

EXAMPLE 2.3. Let $X = [0,1]$ with the Euclidean metric $d$. Define $A$ and $B : X \to X$ by

$$Ax = \frac{1}{2}x \quad \text{and} \quad Bx = \frac{x}{2+x}$$

for all $x$ in $X$. Then $A$ and $B$ are weakly commuting mappings on $X$, but they are not commuting at $x(\neq 0)$ in $X$

DEFINITION 2.4. Let $A$ and $B$ be mappings from a metric space $(X,d)$ into itself. Then $A$ and $B$ are said to be ***compatible mappings*** on $X$ if $\lim_{n\to\infty} d(ABx_n, BAx_n) = 0$ when $\{x_n\}$ is a sequence in $X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$ for some $t$ in $X$.

Obviously, weakly commuting mappings are compatible, but the converse is not necessarily true as in the following example:

EXAMPLE 2.5. Let $X = (-\infty, \infty)$ with the Euclidean metric $d$. Define $A$ and $B : X \to X$ by

$$Ax = x^3 \quad \text{and} \quad Bx = 2 - x$$

for all $x$ in $X$. Since $d(Ax_n, Bx_n) = |x_n - 1|\,|x_n^2 + x_n + 2| \;\to\; 0$ iff $x_n \to 1$,

$$\lim_{n\to\infty} d(BAx_n, ABx_n) = \lim_{n\to\infty} 6\,|x_n - 1|^2 = 0 \quad \text{as } x_n \to 1.$$

Thus, $A$ and $B$ are compatible on $X$, but they are not weakly commuting mappings at $x(=0)$ in $X$. Thus, commutings are also compatible, but the converse is not necessarily true.

We need the following lemmas for our main theorems:

LEMMA 2.6. *Let $A$ and $B$ be compatible mappings from a metric space $(X, d)$ into itself. Suppose that $At = Bt$ for some $t$ in $X$. Then $d(ABt, BAt) = 0$.*

LEMMA 2.7. *Let $A$ and $B$ be compatible mappings from a metric space $(X, d)$ into itself. Suppose that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$ for some $t$ in $X$. Then $\lim_{n\to\infty} BAx_n = At$ if $A$ is continuous.*

## 3. Fixed point theorems

Drawing inspiration from the contractive condition of G.E. Hardy and T.D. Rogers ([5]), we give the following theorems:

THEOREM 3.1. *Let $A$ and $B$ be mappings from a complete metric space $(X, d)$ into itself satisfying the following condition:*

$$\begin{aligned} d(Ax, By) \le\; & a_1\, d(Ax, x) + a_2\, d(By, y) \\ & + a_3\, d(Ax, y) + a_4\, d(By, x) + a_5\, d(x, y) \end{aligned} \tag{3.1}$$

*for all $x, y$ in $X$, where $a_i = a_i(x, y)$, $i = 1, 2, \cdots, 5$ are non-negative functions with*

$$\max\{\sup_{x,y\in X}(a_1 + a_3),\ \sup_{x,y\in X}(a_2 + a_4),\ \sup_{x,y\in X}(a_3 + a_4 + a_5)\} < 1. \tag{3.2}$$

*Suppose that the pair $(A, B)$ is asymptotically regular at some point in $X$. Then $A$ and $B$ have a unique common fixed point in $X$.*

REMARK 3.2. If we replace the condition (3.2) by the following condition:

$$\sup_{x,y\in X}(a_1 + a_2 + a_3 + a_4 + a_5) < 1, \tag{3.3}$$

then the conclusion of Theorem 3.1 is still true.

REMARK 3.3. If we put $A = B$, $a_1 = a_2$ and $a_3 = a_4$ in Theorem 3.1, then we obtain the result of M.D. Guay and K.L. Singh ([4]).

REMARK 3.4. From Remark 3.2, we also obtain the result of D.E. Anderson, K.L. Singh and J.H.M. Whitfield ([1]) (cf. Remark 3.13).

We also refer to the papers of G. Emmanuele ([3]) and M.R. Tasković ([19]) for the existence of fixed points of asymptotically regular mappings under contractive conditions different from the condition (3.1).

Now, let $A$, $B$, $S$ and $T$ be mappings from a metric space $(X, d)$ into itself satisfying the following condition:

$$A(X) \subset T(X) \quad \text{and} \quad B(X) \subset S(X). \tag{3.4}$$

Then for an arbitrary point $x_0$ in $X$, since $A(X) \subset T(X)$, we can choose a point $x_1$ in $X$ such that $Tx_1 = Ax_0$. For this point $x_1$, there exists a point $x_2$ in $X$ such that $Sx_2 = Bx_1$ and so on. Inductively, we can define a sequence $\{y_n\}$ in $X$ such that

$$y_{2n+1} = Tx_{2n+1} = Ax_{2n} \quad \text{and} \quad y_{2n} = Sx_{2n} = Bx_{2n-1} \tag{3.5}$$

for $n = 0, 1, 2, \cdots$.

DEFINITION 3.5. Let $A$, $B$, $S$ and $T$ be mappings from a metric space $(X, d)$ into itself. Then the pair $(A, B)$ is said to be *asymptotically regular* at $x_0$ in $X$ with respect to $S$ and $T$ if

$$\lim_{n \to \infty} d(y_n, y_{n+1}) = 0,$$

where $\{y_n\}$ is the sequence in $X$ defined by (3.5).

If $S$ and $T$ are the identity mappings on $X$, then Definition 3.5 induces to Definition 2.1.

THEOREM 3.6. *Let $A$, $B$, $S$ and $T$ be mappings from a complete metric space $(X, d)$ into itself satisfying the conditions* (3.4), *and* (3.6):

$$\begin{aligned} d(Ax, By) \le\ & a_1\, d(Ax, Sx) + a_2\, d(By, Ty) \\ & + a_3\, d(Ax, Ty) + a_4\, d(By, Sx) + a_5\, d(Sx, Ty) \end{aligned} \tag{3.6}$$

*for all $x, y$ in $X$, where $a_i = a_i(x, y)$, $i = 1, 2, \cdots, 5$ are non-negative functions with the condition (3.2). Suppose that*
(3.7) *one of $A, B, S$ and $T$ is continuous,*
(3.8) *the pairs $A, S$ and $B, T$ are compatible, and*
(3.9) *the pair $(A, B)$ is asymptotically regular at some point in $X$ with respect to $S$ and $T$.*
*Then $A$, $B$, $S$ and $T$ have a unique common fixed point in $X$.*

*Proof.* Let the pair $(A, B)$ be asymptotically regular at $x_0$ in $X$ with respect to $S$ and $T$. Consider the sequence $\{y_n\}$ defined by (3.5). Then we have $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$. Now we prove that $\{y_n\}$ is a Cauchy sequence in $X$. By (3.5) and (3.6), we have

$$\begin{aligned} d(y_{2m}, y_{2n}) &\le d(y_{2m}, Ax_{2m}) + d(Ax_{2m}, Bx_{2n+1}) \\ &\quad + d(Bx_{2n+1}, y_{2n+1}) + d(y_{2n+1}, y_{2n}) \\ &\le d(y_{2m}, y_{2m+1}) + a_1\, d(Ax_{2m}, Sx_{2m}) \\ &\quad + a_2\, d(Bx_{2n+1}, Tx_{2n+1}) \\ &\quad + a_3\, [d(Ax_{2m}, y_{2m}) + d(y_{2m}, y_{2n}) + d(y_{2n}, Tx_{2n+1})] \\ &\quad + a_4\, [d(Bx_{2n+1}, y_{2n+1}) + d(y_{2n+1}, y_{2n}) \\ &\qquad + d(y_{2n}, Sx_{2m})] \\ &\quad + a_5\, [d(Sx_{2m}, y_{2n}) + d(y_{2n}, Tx_{2n+1})] \\ &\quad + d(y_{2n+2}, y_{2n+1}) + d(y_{2n+1}, y_{2n}), \end{aligned}$$

where $a_i = a_i(x_{2m}, x_{2n+1})$, $i = 1, 2, \cdots, 5$. Therefore, we obtain

$$\begin{aligned} (1 - a_3 - a_4 - a_5)\, d(y_{2m}, y_{2n}) \le (1 + a_1 + a_3)\, d(y_{2m}, y_{2m+1}) \\ + (1 + a_2 + a_4)\, d(y_{2n+1}, y_{2n+2}) \\ + (1 + a_3 + a_4 + a_5)\, d(y_{2n}, y_{2n+1}). \end{aligned}$$

From (3.2) and (3.9), by taking the limit as $m$, $n \to \infty$, we show that $\{y_n\}$ is a Cauchy sequence, and hence it converges to some point $z$ in $X$. Consequently, the subsequences $\{Ax_{2n}\}$, $\{Sx_{2n}\}$, $\{Bx_{2n-1}\}$ and $\{Tx_{2n-1}\}$ of $\{y_n\}$ also converge to $z$.

Now, suppose that $S$ is continuous. Since $A$ and $S$ are compatible, Lemma 2.7 implies

$$S^2 x_{2n} \quad \text{and} \quad ASx_{2n} \;\to\; Sz \quad \text{as } n \to \infty.$$

From (3.6) with $a_i = a_i(Sx_{2n}, x_{2n-1})$, $i = 1, 2, \cdots, 5$. we obtain

$$\begin{aligned} d(ASx_{2n}, Bx_{2n-1}) \le (a_1 + a_2) \max\{&d(ASx_{2n}, S^2x_{2n}), \\ &d(Bx_{2n-1}, Tx_{2n-1})\} \\ + (a_3 + a_4 + a_5) \max\{&d(ASx_{2n}, Tx_{2n-1}), \\ &d(Bx_{2n-1}, S^2x_{2n}), d(S^2x_{2n}, Tx_{2n-1})\}. \end{aligned}$$

By taking the limsup, we have

$$d(Sz, z) \le \sup_{x,y \in X} (a_3 + a_4 + a_5)\, d(Sz, z)$$

which, by (3.2), implies $z = Sz$. Again, from (3.6) with $a_i = a_i(z, x_{2n-1})$, $i = 1, 2, \cdots, 5$, we also obtain

$$\begin{aligned} d(Az, Bx_{2n-1}) \le (a_1 + a_3) \max\{&d(Az, Sz), d(Az, Tx_{2n-1})\} \\ + (a_2 + a_4 + a_5) \max\{&d(Bx_{2n-1}, Tx_{2n-1}), \\ &d(Bx_{2n-1}, Sz), d(Sz, Tx_{2n-1})\}. \end{aligned}$$

By taking the limsup, we have

$$\begin{aligned} d(Az, z) &\le (a_1 + a_3) \max\{d(Az, Sz), d(Az, z)\} \\ &\quad + (a_2 + a_4 + a_5)\, d(z, Sz) \\ &\le \sup_{x,y \in X} (a_1 + a_3)\, d(Az, z), \end{aligned}$$

which implies $Az = z$. Since $A(X) \subset T(X)$, $z \in T(X)$ and so there exists a point $u$ in $X$ such that $z = Az = Tu$. Hence, it follows that

$$\begin{aligned} d(z, Bu) &= d(Az, Bu) \\ &\le a_1\, d(Az, Sz) + a_2\, d(Bu, Tu) \\ &\quad + a_3\, d(Az, Tu) + a_4\, d(Bu, Sz) + a_5\, d(Sz, Tu) \\ &\le \sup_{x,y \in X} (a_2 + a_4)\, d(z, Bu), \end{aligned}$$

where $a_i = a_i(z, u)$, $i = 1, 2, \cdots, 5$, which implies $z = Bu$. Since $B$ and $T$ are compatible and $Tu = Bu = z$, $d(TBu, BTu) = 0$ by Lemma 2.6 and hence $Tz = TBu = BTu = Bz$. Moreover, by (3.6), we have

$$\begin{aligned} d(z, Tz) &= d(Az, Bz) \\ &\le \sup_{x,y \in X} (a_3 + a_4 + a_5)\, d(z, Bz), \end{aligned}$$

where $a_i = a_i(z, z)$, $i = 1, 2, \cdots, 5$, so that $z = Tz$. Therefore, $z$ is a common fixed point of $A$, $B$, $S$ and $T$. Similarly, we can also complete the proof when $T$ is continuous.

Next, suppose that $A$ is continuous. Since $A$ and $S$ are compatible, it follows from Lemma 2.7 that

$$A^2 x_{2n} \quad \text{and} \quad SAx_{2n} \rightarrow Az \quad \text{as } n \rightarrow \infty.$$

From (3.6) with $a_i = a_i(Ax_{2n}, x_{2n-1})$, $i = 1, 2, \cdots, 5$, we have

$$\begin{aligned} d(A^2 x_{2n}, Bx_{2n-1}) \leq (a_1 + a_2) &\max\{d(A^2 x_{2n}, SAx_{2n}), \\ &d(Bx_{2n-1}, Tx_{2n-1})\} \\ &+ (a_3 + a_4 + a_5) \max\{d(A^2 x_{2n}, Tx_{2n-1}), \\ &d(Bx_{2n-1}, SAx_{2n}), d(SAx_{2n}, Tx_{2n-1})\}. \end{aligned}$$

By taking the limsup, we have

$$d(Az, z) \leq \sup_{x,y \in X} (a_3 + a_4 + a_5) \, d(z, Az),$$

so that $z = Az$. Hence, since $A(X) \subset T(X)$, there exists a point $v$ in $X$ such that $z = Az = Tv$. Thus we have

$$\begin{aligned} d(A^2 x_{2n}, Bv) \leq (a_1 + a_3 + a_5) &\max\{d(A^2 x_{2n}, SAx_{2n}), \\ &d(A^2 x_{2n}, Tv), d(SAx_{2n}, Tv)\} \\ &+ (a_2 + a_4) \max\{d(Bv, Tv), d(Bv, SAx_{2n})\}, \end{aligned}$$

where $a_i = a_i(Ax_{2n}, v)$, $i = 1, 2, \cdots, 5$. By taking the limsup, we have

$$d(z, Bv) \leq \sup_{x,y \in X} (a_2 + a_4) \, d(z, Bv),$$

which implies that $z = Bv$. Since $B$ and $T$ are compatible and $Tv = Bv = z$, $d(TBv, BTv) = 0$ and hence $Tz = TBv = BTv = Bz$. Moreover, from (3.6) with $a_i = a_i(x_{2n}, z)$, $i = 1, 2, \cdots, 5$, we have

$$\begin{aligned} d(Ax_{2n}, Bz) \leq (a_1 + a_2) &\max\{d(Ax_{2n}, Sx_{2n}), d(Bz, Tz)\} \\ &+ (a_3 + a_4 + a_5) \max\{d(Ax_{2n}, Tz), \\ &d(Bz, Sx_{2n}), d(Sx_{2n}, Tz)\}. \end{aligned}$$

By taking the limsup, we have

$$d(z,Bz) \le \sup_{x,y\in X} (a_3+a_4+a_5)\max\{d(z,Tz), d(Bz,z), d(z,Tz)\},$$

so that $z = Bz$. Since $B(X) \subset S(X)$, there exists a point $w$ in $X$ such that $z = Bz = Sw$ and so we have

$$\begin{aligned} d(Aw,z) &= d(Aw,Bz) \\ &\le (a_1+a_3)\max\{d(Aw,Sw), d(Aw,Tz)\} \\ &\quad + (a_2+a_4+a_5)\max\{d(Bz,Sw), d(Sw,Tz)\}, \end{aligned}$$

where $a_i = a_i(w,z)$, $i = 1,2,\cdots,5$, and

$$d(Aw,z) \le \sup_{x,y\in X} (a_1+a_3)\, d(Aw,z),$$

so that $z = Aw$. Since $A$ and $S$ are compatible and $Aw = Sw = z$, $d(SAw, ASw) = 0$ and hence $Sz = SAw = ASw = Az$. Therefore, $z$ is a common fixed point of $A$, $B$, $S$ and $T$. Similarly, we can also complete the proof when $B$ is continuous.

Finally, it follows easily from (3.6) that the point $z$ is a unique common fixed point of $A$, $B$, $S$ and $T$. This completes the proof.

REMARK 3.7. If $S$ and $T$ are the identity mappings on $X$, then Theorem 3.6 induces to Theorem 3.1.

REMARK 3.8. The condition of asymptotic regularity is necessary in Theorem 3.6 as in the following example:

EXAMPLE 3.9. Let $X = [1,\infty)$ with the Euclidean metric $d$. Define $A = B$ and $S = T : X \to X$ by

$$Ax = 2x \quad \text{and} \quad Sx = 4x$$

for all $x$ in $X$. Then the conditions (3.4) and (3.6) are satisfied with $a_i = 0$, $i = 1,2,3,4$ and $a_5 = c$, where $\frac{1}{2} \le c < 1$. The other hypotheses of Theorem 3.6 are satisfied except the condition (3.9). Indeed, for the sequence $\{y_n\}$ defined by (3.5),

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0 \quad \text{iff} \quad x_0 = 0$$

but 0 does not belong to $X$. Here, none of mappings has a fixed point in $X$.

REMARK 3.10. If we replace the condition (3.2) by the following condition:

$$k \equiv \sup_{x,y\in X} (a_1 + a_2 + 2\,a_3 + 2\,a_4 + a_5) < 1,$$

then the conclusion of Theorem 3.6 is still true except the condition of asymptotic regularity. Indeed, let $\{y_n\}$ is a sequence in $X$ defined (3.5). From (3.6), we have

$$\begin{aligned} d(y_{2n+1}, y_{2n+2}) &= d(Ax_{2n}, Bx_{2n+1}) \\ &\leq a_1\, d(y_{2n}, y_{2n+1}) + a_2\, d(y_{2n+1}, y_{2n+2}) \\ &\quad + a_4\, [\, d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})\,] \\ &\quad + a_5\, d(y_{2n}, y_{2n+1}), \end{aligned}$$

where $a_i = a_i(x_{2n}, x_{2n+1})$, $i = 1, 2, \cdots, 5$. In the above inequality, if $d(y_{2n+1}, y_{2n+2}) > d(y_{2n}, y_{2n+1})$, then we have

$$d(y_{2n+1}, y_{2n+2}) \leq (a_1 + a_2 + 2\,a_4 + a_5)\, d(y_{2n+1}, y_{2n+2}),$$

which is a contradiction. Thus, we have

$$d(y_{2n+1}, y_{2n+2}) \leq k\, d(y_{2n}, y_{2n+1}).$$

Similarly, we obtain

$$d(y_{2n}, y_{2n+1}) \leq k\, d(y_{2n-1}, y_{2n})$$

It follows from the above facts that

$$d(y_n, y_{n+1}) \leq k\, d(y_{n-1}, y_n).$$

Since $k < 1$, by Lemma of G. Jungck ([6]), $\{y_n\}$ is a Cauchy sequence in $X$. As in the proof Theorem 3.6, this conclusion follows easily.

REMARK 3.11. If we put $A = B$ and $S = T$ in Theorem 3.6, by Remark 3.10, we obtain easily the result of R.N. Mukherjee ([14]) by replacing commutativity by compatibility.

H. Kaneko and S. Sessa ([11]) proved the following thorem, which extended the result of K.M. Das and K.V. Naik ([2]):

**THEOREM 3.12.** *Let $A$ and $S$ be mappings from a complete metric space $(X,d)$ into itself satisfying the following conditions:*

$$A(X) \subset S(X), \tag{3.10}$$

$$\begin{aligned} d(Ax,Ay) \le h \max\{&d(Ax,Sx), d(Ay,Sy),\\ &d(Ax,Sy), d(Ay,Sx), d(Sx,Sy)\} \end{aligned} \tag{3.11}$$

*for all $x, y$ in $X$, where $0 \le h < 1$. Suppose that one of $A$ and $S$ is continuous, and the pair $A, S$ is compatible.*
*Then $A$ and $S$ have a unique common fixed point in $X$.*

**REMARK 3.13.** As pointed out by S. Massa ([13]), the condition (3.11) is equivalent to the following condition:

$$\begin{aligned} d(Ax,Ay) \le\ & a_1\, d(Ax,Sx) + a_2\, d(Ay,Sy)\\ &+ a_3\, d(Ax,Sy) + a_4\, d(Ay,Sx) + a_5\, d(Sx,Sy) \end{aligned} \tag{3.12}$$

for all $x, y$ in $X$, where $a_i = a_i(x,y)$, $i = 1, 2, \cdots, 5$ are non-negative functions with (3.3). Clearly, the condition (3.12) is obtained from (3.6) with $A = B$ and $S = T$. Since $A(X) \subset S(X)$ by (3.10), for an arbitrary point $x_0$ in $X$, we can choose a point $x_1$ in $X$ such that $y_1 = Sx_1 = Ax_0$. Continuing in this manner, we can define a sequence $\{y_n\}$ as follows:

$$y_n = Sx_n = Ax_{n-1} \tag{3.13}$$

for $n = 1, 2, 3, \cdots$. K.M. Das and K.V. Naik ([2]) proved that the sequence $\{y_n\}$ defined by (3.13) converges to some point $z$ in $X$ and $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$. Thus, the condition (3.9) can be dropped. Therefore, Theorem 3.6 is a generalization of Theorem 3.12.

**THEOREM 3.14.** *Let $A$, $B$, $S$ and $T$ be mappings from a complete metric space $(X,d)$ into itself satisfying the conditions* (3.4), (3.7), (3.8), *and* (3.14)*:*

$$\begin{aligned} d(Ax,By) \le h \max\{&d(Ax,Sx), d(By,Ty),\\ &\tfrac{1}{2}[\,d(Ax,Ty) + d(By,Sx)\,], d(Sx,Ty)\} \end{aligned} \tag{3.14}$$

for all $x, y$ in $X$, where $0 \leq h < 1$.
*Then $A$, $B$, $S$ and $T$ have a unique common fixed point in $X$.*

*Proof.* In virtue of Lemma 3.1 in [8], if we replace the condition (3.6) by (3.14), then the condition of asymptotic regularity can be dropped. As in the proof of Theorem 3.6, this conclusion follows easily.

REMARK 3.15. Our main results in this paper generalize and improve a number of fixed point theorems for commuting mappings ([2], [6], [12] and [18]).

REMARK 3.16. Theorem 3.14 generalizes the result of G. Jungck ([8]) by using any one continuous mapping in stead of the continuity of both $S$ and $T$. If we do not assume that any one of mappings is continuous, Theorem 3.14 is no longer true as in the following example:

EXAMPLE 3.17. Let $X = [0, 1]$ with the Euclidean metric $d$. Define $A = B$ and $S = T : X \to X$ by

$$Ax = \begin{cases} \frac{1}{8}, & \text{if } x = 0 \\ \frac{1}{8}x, & \text{if } x \neq 0 \end{cases} \quad \text{and} \quad Sx = \begin{cases} 1, & \text{if } x = 0 \\ \frac{1}{2}x, & \text{if } x \neq 0 \end{cases}$$

for all $x$ in $X$. $A(X) = (0, \frac{1}{8}] \subset (0, \frac{1}{2}] \subset S(X)$. Moreover, we obtain

$$d(AS0, SA0) = \frac{1}{16} < \frac{7}{8} = d(S0, A0)$$

and $ASx = SAx = \frac{1}{16}x$ for all $x$ in $X - \{0\}$. So, $A$ and $S$ are compatible on $X$. Further, we obtain

$$\begin{aligned} d(Ax, Ay) &= \begin{cases} 0, & \text{if } x = y = 0 \\ \frac{1}{8}(1 - x) < \frac{1}{4}(1 - \frac{1}{2}x) = \frac{1}{4}d(Sx, Sy), & \text{if } x > y = 0 \\ \frac{1}{8}(1 - y) < \frac{1}{4}(1 - \frac{1}{2}y) = \frac{1}{4}d(Sx, Sy), & \text{if } y > x = 0 \\ \frac{1}{8}|x - y| = \frac{1}{4}d(Sx, Sy), & \text{if } x, y \neq 0, \end{cases} \\ &\leq \frac{1}{4}\max\{d(Ax, Sx), d(Ay, Sy), \\ &\qquad \frac{1}{2}[d(Ax, Sy) + d(Ay, Sx)], d(Sx, Sy)\} \end{aligned}$$

for all $x, y$ in $X$. We find that all the hypotheses of Theorem 3.14 is satisfied except the continuity of $A$ and $S$, but none of the given mappings has a fixed point in $X$.

REMARK 3.18. Quite recently, in [10], G. Jungck, P.P. Murthy and Y.J. Cho introduced the concept of compatible mappings of type $(A)$ in metric spaces and obtained some relations between compatible mappings and compatible mappings of type $(A)$. Also, they proved that two concepts of compatibility and compatibility of type $(A)$ are equivalent under some conditions. If the given mappings in all the theorems of this section are continuous, all the theorems in this paper are still true even though the condition of the compatibility is replaced by the compatibility of type $(A)$.

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