Pusan Kyŏngnam Math. J. 8(1992), No. 2, pp 151–162

ON THE RELATION BETWEEN ORDERED PROPERTIES IN REGULAR NEAR–RINGS AND AUTOMORPHISM GROUPS IN LAMINATED NEAR–RINGS

Yong Uk Cho

1. Introduction

Let S be a semigroup and fix an element $a \in S$. Define a new multiplication $*$ on S by $\mathbf{x} * y = \mathbf{x}\ a\ y$ for all $\mathbf{x}, y \in S$. $(S, *)$ is called as a laminated semigroup of S with laminating element (or laminator) a, which is denoted by S_a. The original semigroup S is referred to as the base semigroup. Similarly, if N is any near-ring, we can form another near-ring by defining addition to coincide with the addition in N but defining $\mathbf{x} * y = \mathbf{x}\ a\ y$ for all $\mathbf{x}, y \in N$ as multiplication, where a is some fixed element of N. The new near-ring is a laminated near-ring with laminating element (or laminator) a, and is denoted by N_a. In [7] K.D. Magill initiated his study of the automorphism group of a certain class of near-rings, and laminated near-rings of continuous selfmaps of topological group were introduced. Specially in [8], let p be any complex polynomial and let $\mathcal{N}_p$ denote the near-ring of all continuous selfmaps of the complex plane where addition of functions is pointwise and the product $f\ g$ of two functions f and g in $\mathcal{N}_p$ is defined by $f \circ p \circ g$, where $\circ$ is a composition. The near-ring $\mathcal{N}_p$ is called as a laminated near-ring with laminating element p. Further study of laminated near-ring were in [1] and [9].

G. Mason ([11]) studied the notion of regularity of near-rings. He proved that for any zero symmetric near-ring with identity, the concepts of left regularity, strong left regularity and strong right regularity

Received October 5, 1992

This paper was supported (in part) by NON DIRECTED RESEARCH FUND, Korea Research Foundation, 1991

of near-rings are equivalent. In [2] these equivalent conditions of regularity are slightly weakened and added another regularity of near-rings. In [11], ordered near-rings are introduced.

In section 2, we will introduce more generalized concepts of laminated near-rings in the class of continuous functions form topological space into topological group and study some relation between topological properties and algebraic properties. In section 3, we will construct another generalized notion of laminated near-ring in the class of mappings from topological space into any near-ring and define ordered relation on this new near-ring. Consequently, from arbitrary ordered near-ring, one obtain new concrete ordered near-ring which is a representation of ordered near-ring. Next, we will investigate some ordered properties of regular near-rings namely, introduce the concept of positive derivation on an ordered near-ring and get their properties on a class of ordered left regular near-rings. Finally, we will obtain some results in automorphism groups of arbitrary near-rings which are more important in the study of algebraic research and one will apply these automorphism groups to laminated near-ring and more generalized concept of laminated near-ring, for example sandwich near-ring in one's other papers.

2. Sandwich near-rings of continuous functions

First, we will give a more general concept of laminated near-rings of near-rings of continuous selfmaps.

Let X be a topological space, G an additive topological group and α a continuous function from G into X. Denote by $N(X, G, \alpha)$ the near-ring of all continuous functions from X into G where addition of functions is pointwise and the product $f\ g$ of two functions f, g in $N(X, G, \alpha)$ is defined by $f\ g = f \circ \alpha \circ g$ where the operation $\circ$ is composition of functions. The near-ring $N(X, G, \alpha)$ is referred to as a sandwich near-ring with sandwich function α. Similarly, let X and G be topological spaces and denote by $S(X, G, \alpha)$ the semigroup of all continuous functions from X into G with multiplication defined as before. If $G = X$ and α is identity map then $N(X, G, \alpha)$ becomes the near-ring of all continuous selfmaps of G under pointwise addition and ordinary composition. In this case, we use the simpler notation $N(G)$.

Similarly, if $G = X$ is topological space and α is identity map then $S(X, G, \alpha)$ is denoted by $S(G)$.

THEOREM 2.1. *Let $N(X, G, \alpha)$ and $N(Y, H, \beta)$ be two sandwich near-rings. If h is a homeomorphism from X onto Y and t is a topological isomorphism from G onto H such that $h \circ \alpha = \beta \circ t$, then the mapping ϕ from $N(X, G, \alpha)$ into $N(Y, H, \beta)$ defined by*

$$\phi(f) = f \circ f \circ h^{-1} \quad \text{for each } f \text{ in } N(X, G, \alpha),$$

is an near-ring isomorphism.

Proof. Consider the diagram :

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & G \arrow[r, "\alpha"] \arrow[d, "t"'] & X \arrow[d, "h"'] \\ Y \arrow[r, "\phi(f)"] & H \arrow[r, "\beta"] & Y \end{tikzcd} \qquad \text{and} \qquad h \circ \alpha = \beta \circ t$$

Clearly, we see that the diagram is commutative. To show that ϕ is a near-ring isomorphism, for any f and g in $N(X, G, \alpha)$, we have

$$\phi(f + g) = t \circ (f + g) \circ h^{-1} = t \circ f \circ h^{-1} + t \circ g \circ h^{-1} = \phi(f) + \phi(g)$$

and

$$\begin{aligned} \phi(f\ g) &= \phi(f \circ \alpha \circ g) = t \circ f \circ \alpha \circ g \circ h^{-1} \\ &= t \circ f \circ (h^{-1} \circ h) \circ \alpha \circ (t^{-1} \circ t) \circ g \circ h^{-1} \\ &= (t \circ f \circ h^{-1})(h \circ \alpha \circ t^{-1})(t \circ g \circ h^{-1}) \\ &= \phi(f) \circ \beta \circ \phi(g) = \phi(f) \circ \phi(g) \end{aligned}$$

Next we must show that ϕ is injective : Suppose $\phi(f) = \phi(g)$, where f and g are in $N(X, G, \alpha)$. Then by definition of hypothesis, $t \circ f \circ h^{-1} = t \circ g \circ h^{-1}$, since t is a topological isomorphism and h is a homeomorphism, we obtain that $f = g$ so that ϕ is injective. Finally, to show that ϕ is surjective : For any continuous function k from Y into H, we construct some continuous function $f = t^{-1} \circ k \circ h$ from X into G, so that we get that

$$\phi(f) = \phi(t^{-1} \circ k \circ h) = t \circ t^{-1} \circ k \circ h \circ h^{-1} = k.$$

Consequently, the proof of the theorem is complete.

The isomorphism ϕ of above theorem is called natural. In this section we will discuss the following problem :

PROBLEM. Find reasonable conditions on the spaces, groups and sandwich functions which will insure that every isomorphism from $N(X,G,\alpha)$ onto $N(X,G,\beta)$ is natural.

Most of the results about near-rings that we obtain in this section are consequences of analogous results about the multiplicative semigroups of these near-rings. We denote the multiplicative semigroup of $N(X,G,\alpha)$ by $S(X,G,\alpha)$. The product $f\ g$ of two elements f,g in $S(X,G,\alpha)$ is, of course, given by $f\ g = f \circ \alpha \circ g$. Furthermore, when we write $S(X,G,\alpha)$ we assume that G is a topological space and not necessarily a topological group. We need to introduce some notation : For any point $a \in G$, the symbol $< a >$ will denote the constant function in $S(X,G,\alpha)$ (or $N(X,G,\alpha)$) which maps all of X into the point a.

THEOREM 2.2. *Let ϕ be an isomorphism from $S(X,G,\alpha)$ onto $S(Y,H,\beta)$. Then there exists a unique bijection t from G onto H such that $\phi < a > = < t(a) >$ for each a in G.*

Proof. We first need to show that the constant functions from X into G are precisely the left zeros of the semigroup $S(X,G,\alpha)$. Indeed, it is immediate that every constant function is a left zero, for any constant function $< a >$ from X into G and for each f in $S(X,G,\alpha)$, we calculate that

$$(< a > f)(\mathbf{x}) = (< a > \circ\alpha \circ f)(\mathbf{x}) = < a > [\alpha\{f(\mathbf{x})\}] = a = < a > (\mathbf{x})$$

for each $\mathbf{x} \in X$. Thus we see that $< a > f = < a >$, that is $< a >$ is a left zero.

Conversely, suppose that f is a left zero of $S(X,G,\alpha)$, that is $f\ g = f$ for all g in $S(X,G,\alpha)$. Fix any a in G and let $\mathbf{x}$ in X be given. We then have

$$f(\mathbf{x}) = (f < a >)(\mathbf{x}) = (f \circ \alpha \circ < a >)(\mathbf{x}) = < f\{\alpha(a)\} > (\mathbf{x}).$$

In other words, $f = < f\{\alpha(a)\} >$, which is a constant function from X into G.

Now let a be any arbitrary element of G. Since ϕ sends left zeros of $S(X,G,\alpha)$ to left zeros of $S(Y,H,\beta)$, there exists an element $w \in H$

such that $\phi < a > = < w >$. From this fact, we can define a function $t : G \to H$ which is defined by $t(a) = w$, for all a in G. The function t is a bijection : Consider the following diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & G \arrow[r, "\alpha"] \arrow[d, "t"'] & X \arrow[d, "h"] \\ Y \arrow[r, "\phi(f)"] & H \arrow[r, "\beta"] & Y \end{tikzcd}$$

Assume that $t(a) = t(b)$ $(a, b \in G)$. Then $< t(a) > = < t(b) >$. By definition we have $\phi < a > = \phi < b >$, then since ϕ is a bijection, $< a > = < b >$, that is, $a = b$. Next, let c be any element of H. Then $< c >: Y \to H$ is a constant function which is also a left zero. Since ϕ is a bijection, there exists a in G such that the constant function $< a >: X \to G$ is left zero and we have $\phi < a > = < c >$, from this, we get $t(a) = c$. Whence the function t is a bijection. It is immediate that $\phi < a > = < t(a) >$ and the uniqueness of t is easily verified.

In the case of near-rings, we can say a little more.

THEOREM 2.3. *Let ϕ be an isomorphism from $N(X, G, \alpha)$ onto $N(Y, H, \beta)$. Then there exists a unique isomorphism t (not necessarily a topological isomorphism) from G onto H such that $\phi < a > = < t(a) >$ for each a in G.*

Proof. In view of Theorem 2.2, we need only show that t is a group homomorphism. Let $a, b \in G$ be given. Using the fact that ϕ is a group isomorphism, we get

$$\begin{aligned} < t(a+b) > &= \phi < a+b > = \phi(< a > + < b >) = \phi < a > + \phi < b > \\ &= < t(a) > + < t(b) > = < t(a) + t(b) > \end{aligned}$$

This means that the contant functions $< t(a+b) >$ and $< t(a) + t(b) >$ coincide, by the defintion of these constant functions we see that $t(a + b) = t(a) + t(b)$. Thus the proof of theorem is complete.

REMARK 2.4. Let X be a connected topological space and G a totally disconnected group (or space). For purposes of discussion, we will refer to the corresponding sandwich semigroup $S(X, G, \alpha)$ as a meager semigroup and the corresponding sandwich near-ring $N(X, G, \alpha)$ as a meager near-ring. One easily sees that $S(X, G, \alpha)$, in this case, consists entirely of constant functions and as a left zero semigroup. As for the meager near-rings $N(X, G, \alpha)$ and $N(Y, H, \beta)$, they will be isomorphic if and only if G and H are algebraically isomorphic : Indeed, if t is any algebraic isomorphism from G onto H, then the mapping ϕ defined by $\phi < a >=< t(a) >$ is an isomorphism from $N(X, G, \alpha)$ onto $N(Y, H, \beta)$, in fact,

$$\begin{aligned}\phi(< a > + < b >) &= \phi < a + b >=< t(a+b) >\\ &=< t(a) + t(b) >=< t(a) > + < t(b) >\\ &= \phi < a > + \phi < b >,\end{aligned}$$

and

$$\phi(< a >< b >) = \phi(< a > \circ\alpha\circ < b >) = \phi < a >=< t(a) >,$$

on the other hand

$$\begin{aligned}\phi < a > \phi < b > &= \phi < a > \circ\beta\circ \phi < b >\\ &=< t(a) > \circ\beta\circ < t(b) >\\ &=< t(a) >,\end{aligned}$$

so that one see that $\phi(< a >< b >) = \phi < a > \phi < b >$. Next the mapping ϕ is bijective. Indeed, if $\phi < a >= \phi < b >$, then $< t(a) >=< t(b) >$ which implies $t(a) = t(b)$, since t is bijective, $a = b$. Consequently, $< a >=< b >$, and let $w \in H$ such that $< w >: Y \to H$ is a left zero mapping, then since t is surjective, there exists a in G such that $t(a) = w$, it follows that $< w >=< t(a) >= \phi < a >$. Thus ϕ is an isomorphiam of near-rings. Converse is obtained from the same manner of theorem 2.3.

3. Ordered properties of reqular near-rings

A near-ring N is called (partially) ordered provided that $a \geq b$ implies $a + \mathbf{x} \geq b + \mathbf{x}$ and $\mathbf{x} + a \geq \mathbf{x} + b$ for all $\mathbf{x}$ in N, $a \geq 0$ and $b \geq 0$ imlies $a\ b \geq 0$.

The following facts are exident :

$a \geq b$ if and only if $-b + a \geq 0, \quad a - b \geq 0$

$a \geq 0$ if and only if $-a \leq 0$ and

$a \leq r$ and $b \leq s$ implies $a + b \leq r + s$.

Furthermore we see that :

$a \geq 0$ and $-a \geq 0$ if and only if $a = 0$ and

$a \geq 0$ and $b \geq 0$ implies $a + b \geq 0$ and $a\, b \geq 0$.

We construct another types of near-rings of functions on a topological space. Let X be any topological space and let N arbitrary near-ring. Denote by $N(X, N)$ the set of all functions from X into N. We can define two binary operation on $N(X, N)$, addition of functions is pointwise and the product $f\ g$ of two functions f, g in $N(X, N)$ is defined by $(f\ g)(\mathbf{x}) = f(\mathbf{x})g(\mathbf{x})$ for all $\mathbf{x}$ in X. It is obvious that $N(X, N)$ becomes a near-ring. The zero element is the constant function 0, the additive inverse of f in $N(X, N)$ is characterized by the formula $(-f)(\mathbf{x}) = -f(\mathbf{x})$. If N has an identity 1 then the identity element of $N(X, N)$ is the constant function 1, that is, $1(\mathbf{x}) = 1$ for all $\mathbf{x} \in X$. One sees that, for any f in $N(X, N)$, $(f\ 1)(\mathbf{x}) = f(\mathbf{x})1(\mathbf{x}) = f(\mathbf{x})1 = f(\mathbf{x})$. Similarly $(1\ f)(\mathbf{x}) = f(\mathbf{x})$. So, the constant function 1 is the identity of $N(X, N)$.

THEOREM 3.1. *If N is an ordered near-ring and ordering on $N(X, N)$ is defined by $f \geq g$ if and only if $f(\mathbf{x}) \geq g(\mathbf{x})$ for all $\mathbf{x} \in X$, then $N(X, N)$ is also an ordered near-ring.*

Proof. Straightforwards.

In this section, we will introduce the concept of positive derivation on ordered near-rings and show that on ordered near-ring which is left regular and has the additional property that $a^2 \geq 0$ for each a in N cannot have nontrivial positive derivations. An ordered near-ring which is also integral is called an ordered integral near-ring and which is also a near-field is called an ordered near-field. In an ordered near-ring N, $P = P(N) = \{a \in N | a \geq 0\}$ is called the positive cone of N. Every ordering of a near-ring N is determined by P that is $a \leq b$ if and only if $b - a \in P$ or $-a + b \in P$. A near-ring N is said to be left regular if for each a in N there exists an element $\mathbf{x}$ in N such that $a = \mathbf{x}\ a^2$.

LEMMA 3.2. ([11]) *If N is an ordered near-ring with positive cone P, then P has the following properties :*

(1) *P is a subseminear-ring of N,*

(2) *$a \in P$ and $-a \in P$ implies $a = 0$,*

(3) *$P + a = a + P$ for all a in P.*

Conversely, if a near-ring N has a subset P satisfying these above conditions and we define $a \leq b$ to mean that $b - a \in P$ or *$-a + b \in P$, then N becomes an ordered near-ring with positive cone P.*

LEMMA 3.3. *Let N be an ordered near-ring with positive cone P. Then N is linearly ordered if and only if $N = P \cup (-P)$ where $-P = \{-a \in N | a \in P\}$.*

LEMMA 3.4. ([11] Karel. Maxon, Ligh-Neal) *Let N be a near-field. Then*

(1) *for all $a \in N$, $a^2 = 1$ if and only if $a = 1$ or $a = -1$,*

(2) *for any $a, b \in N$, $a\ (-b) = -(a\ b) = (-a)\ b$.*

The proof of Lemma 3.4 is more difficult and long.

THEOREM 3.5. *In any linearly ordered near-field N, all sequares of non-zero elements are positive.*

Proof. Let a be any non-zero element in N. From Lemma 3.3, either a in P or $-a$ in P. Since P is closed under multiplication, $a^2 \in P$ and $(-a)(-a) = -(a(-a)) = -(-(a^2)) = a^2 \in P$ by using Lemma 3.4. In either case, we are done.

Now we will introduce a positive derivation on ordered near-rings, and investigate ordered properties of ordered regular near-rings.

DEFINITION 3.6. Let N be an ordered near-ring. A selfmap δ of N is called a positive derivation on N, if δ satisfies the following properties :

(1) $\delta(a + b) = \delta(a) + \delta(b)$ for each a, b in N,

(2) $\delta(a\ b) = \delta(a)b + a\delta(b)$ for each a, b in N,

(3) $\delta(a) \geq 0$ for each a in N such that $a \geq 0$.

LEMMA 3.7. ([2]) *For any left regular near-ring N, if a, b in N such that $a\ b = 0$ and $a^n = a\ 0$ for any positive integer $n \geq 2$, then $a = 0$. In particular, if N is zero-symmetric, then N is reduced.*

LEMMA 3.8. ([2]) *Every left regular near-ring is regular. Furthermore, for any $a, \mathbf{x}$ in N such that $a = \mathbf{x}\, a^2$, we have $a\, \mathbf{x} = \mathbf{x}\, a$.*

Before proving the main theorem, we first prove the following, hereafter of this section, we may assume that N is zero-symmetric.

THEOREM 3.9. *Let N be an ordered left regular near-ring with $a^2 \geq 0$ for each a in N. If δ is a positive derivation defined on N and a in N with $a \geq 0$, then $\delta(a) = 0$.*

Proof. Suppose that N is ordered left regular and a is any element of N with $a \geq 0$. Then there exists an element $\mathbf{x}$ in N such that $a = \mathbf{x}\, a^2$. By Lemma 3.8, we get $a\,\mathbf{x} = \mathbf{x}\, a$ and $a = a\,\mathbf{x}\, a$. Applying δ for $a = a\,\mathbf{x}\,a$, we have $\delta(a) = \delta(a\,\mathbf{x}\,a) = \delta(a)\,\mathbf{x}\,a + a\,\delta(\mathbf{x}\,a)$. Multiplying on the right side of this equation by a, $\delta(a)\,a = \delta(a)\,\mathbf{x}\,a^2 + a\,\delta(\mathbf{x}\,a)\,a$. This implies that $a\,\delta(\mathbf{x}\,a)\,a = 0$. Thus $[a\,\delta(\mathbf{x}\,a)]^2 = a\,\delta(\mathbf{x}\,a)\,a\,\delta(\mathbf{x}\,a) = 0$. From Lemma 3.7, N is reduced, so that $a\,\delta(\mathbf{x}\,a) = 0$. Consequently, $\delta(a) = \delta(a)\,\mathbf{x}\,a$.

Next, applying δ for $a\,\mathbf{x} = \mathbf{x}\,a$, $\delta(\mathbf{x}\,a) = \delta(a\,\mathbf{x}) = \delta(a)\,\mathbf{x} + a\,\delta(\mathbf{x})$. Multiplying on the right side by a, $\delta(\mathbf{x}\,a)\,a = \delta(a)\,\mathbf{x}\,a + a\,\delta(\mathbf{x})\,a$. Obviously, $\delta(\mathbf{x}\,a)\,a = 0$, by using Lemma 3.7 and $a\,\delta(\mathbf{x}\,a)\,a = 0$. Thus we obtain the equation $\delta(a)\,\mathbf{x}\,a = -a\,\delta(\mathbf{x})\,a$, together with $\delta(a) = \delta(a)\,\mathbf{x}\,a$, we have $\delta(a) = -a\,\delta(\mathbf{x})\,a$. Since N is reduced, $\mathbf{x}^2\,a \geq 0$ and since δ is positive, $\delta(\mathbf{x}^2\,a) \geq 0$, so that $\delta(\mathbf{x})\,\mathbf{x}\,a + \mathbf{x}\,\delta(\mathbf{x}\,a) \geq 0$. Multiplying on the right by a with $a \geq 0$. It follows that $\delta(\mathbf{x})\,\mathbf{x}\,a^2 + \mathbf{x}\,\delta(\mathbf{x}\,a)\,a = \delta(\mathbf{x})\,a + 0 = \delta(\mathbf{x})\,a \geq 0$. Then from $a \geq 0$ and $a\,\delta(a) = -\delta(a) \geq 0$, we have $\delta(a) \leq 0$. Because δ is positive, consequently $\delta(a) = 0$

THEOREM 3.10. *Let N be an ordered left regular square positive near-ring. If δ is a positive derivation on N then $\delta(\mathbf{x}) = 0$ for each $\mathbf{x}$ in N.*

Proof. From Theorem 3.9.

COROLLARY 3.11. *Let F be a linearly ordered left regular near-field and δ is a positive derivation defined on F. Then $\delta(\mathbf{x}) = 0$ for each $\mathbf{x}$ in F.*

Proof. From Theorem 3.5 and Theorem 3.10.

4. Automorphism groups of any near-rings

In this section, we will consider N is a near-ring with identity 1, and study the group $\mathrm{Aut}(N)$ of all near-ring-automorphisms of N under composition. If $S(N)$ is the symmetric group of N, then $\mathrm{Aut}(N)$ is a subgroup of $S(N)$.

THEOREM 4.1. *Let N be a near-ring with identity 1. If for each a in N_d, a is invertible and α_a is a selfmap of N satisfying $\alpha_a(\mathbf{x}) = a\ \mathbf{x}\ a^{-1}$ for all $\mathbf{x}$ in N, then α_a is in $Aut(N)$.*

Proof. Let $\mathbf{x}, y$ in N,

$$\alpha_a(\mathbf{x}+y) = a\ (\mathbf{x}+y)\ a^{-1} = a\ \mathbf{x}\ a^{-1} + a\ y\ a^{-1} = \alpha_a(\mathbf{x}) + \alpha_a(y)$$

and

$$\begin{aligned}\alpha_a(\mathbf{x}\ y) &= a\ (\mathbf{x}\ y)\ a^{-1} = a\ (\mathbf{x}\ a^{-1}\ a\ y)\ a^{-1}\\ &= (a\ \mathbf{x}\ a^{-1})(a\ y\ a^{-1}) = \alpha_a(\mathbf{x})\alpha_a(y)\end{aligned}$$

so that α_a is a near-ring homomorphism. It is sufficient to show that α_a is bijective. Indeed, if $\alpha_a(\mathbf{x}) = \alpha_a(y)$ then $a\ \mathbf{x}\ a^{-1} = a\ y\ a^{-1}$, since a is invertible, we have $\mathbf{x} = y$. Next, for every y in N there exist $a^{-1}\ y\ a$ in N such that $\alpha_a(a^{-1}\ y\ a) = a\ a^{-1}\ y\ a\ a^{-1} = y$.

We say that α_a is an inner automorphism of N and we have the notation $\mathrm{Inn}(N) = \{\alpha_a | a \in N_d, a : \text{invertible}\}$. Then we obtain the following results.

THEOREM 4.2. *Let N be any near-ring with identity 1. If for every a in N_d, a is invertible and α_a is an inner automorphism of N, then we have the following :*

(1) The mapping $\alpha|K \to Aut(N)$ defined by $\alpha(a) = \alpha_a$ is a group homomorphism, where $K = \{a \in N : a \in N_d, a :$ invertible$\}$,

(2) $Inn(N)$ is a normal subgroup of $Aut(N)$,

(3) $K/Z(N) \cong Inn(N)$, where $Z(N) = \{a \in K | a\ \mathbf{x} = \mathbf{x}\ a$, for all $\mathbf{x} \in N\}$.

Proof. Clearly K is a group under multiplication, so we can define the mapping $\alpha : K \to \mathrm{Aut}(N)$ by $\alpha(a) = \alpha_a$, to show that α is a group homomorphism, that is, $\alpha(a\ b) = \alpha(a)\ \alpha(b)$ for all $a, b \in K$, we need only to show that $\alpha_{a\ b} = \alpha_a\ \alpha_b$: In fact, let $\mathbf{x} \in N$,

$$\begin{aligned}\alpha_{a\ b}(\mathbf{x}) &= (a\ b)\ \mathbf{x}\ (a\ b)^{-1} = a\ (b\ \mathbf{x}\ b^{-1})\ a^{-1} \\ &= a\ (\alpha_b(\mathbf{x}))\ a^{-1} = \alpha_a(\alpha_b(\mathbf{x})) \\ &= (\alpha_a\ \alpha_b)(\mathbf{x}),\end{aligned}$$

hence (1) is proved by using the above Theorem 4.1.

To show (2), for any two α_a, α_b in $\mathrm{Inn}(N)$, we see that $\alpha_a\ \alpha_b = \alpha_{a\ b}$, $(\alpha_a)^{-1} = \alpha_{a^{-1}}$ so that $\mathrm{Inn}(N)$ is a subgroup of the group $\mathrm{Aut}(N)$.

Next, we have to show that normality property : For any $\sigma \in \mathrm{Aut}(N)$ and $\alpha_a \in \mathrm{Inn}(N)$, we have that for each $\mathbf{x} \in N$,

$$\begin{aligned}(\sigma^{-1}\alpha_a\sigma)(\mathbf{x}) &= (\sigma^{-1}\alpha_a)\sigma(\mathbf{x}) = \sigma^{-1}[\alpha_a(\sigma(\mathbf{x}))] \\ &= \sigma^{-1}[a\ (\sigma(\mathbf{x}))\ a^{-1}] = \sigma^{-1}(a)\ \mathbf{x}\ \sigma^{-1}(a^{-1}) \\ &= \sigma^{-1}(a)\ \mathbf{x}\ (\sigma^{-1}(a))^{-1} = \alpha_{\sigma^{-1}(a)}(\mathbf{x}).\end{aligned}$$

Consequently, this follows that $\mathrm{Inn}(N)$ is a normal subgroup of $\mathrm{Aut}(N)$.

(3) One easily verifies that $\mathrm{Inn}(N) = \mathrm{Im}\alpha$. We first show that $\mathrm{Ker}\alpha = Z(N)$. Let $a \in \mathrm{Ker}\alpha$, that is, $\alpha_a = 1_N$ identity mapping of N. Then for each $\mathbf{x} \in N$, $a\ \mathbf{x}\ a^{-1} = \mathbf{x}$. This implies $a\ \mathbf{x} = \mathbf{x}\ a$ for all $\mathbf{x} \in N$, namely $a \in Z(N)$. It follows that $\mathrm{Ker}\alpha \subset Z(N)$. The converse inclusion is proved analogously. Finally, from the first group isomorphism Theorem, we have that $K/Z(N) \cong \mathrm{Inn}(N)$. Therefore the proof is complete.

It is called that $\mathrm{Inn}(N)$ is an inner automorphism group of N.

COROLLARY 4.3. *If N is a commutative near-ring with* 1, *then*

$$\mathit{Inn}(N) = \{1_N\}.$$

Proof. Obviously, we see that $1_N = \alpha_1$. Conversely, let $\alpha_a \in \mathrm{Inn}(N)$. Then for any $\mathbf{x} \in N$,

$$\alpha_a(\mathbf{x}) = a\ \mathbf{x}\ a^{-1} = a\ a^{-1}\ \mathbf{x} = 1\ \mathbf{x} = \mathbf{x} = 1_N(\mathbf{x}),$$

this shows that $\alpha_a = 1_N$.

References

1. D.K. Blevins, K.D. Magill and P.R. Misra, *More on automorphism groups of laminated near-rings*, Proc. Edinburgh Math. Soc. **31** (1988), 185–195.
2. Y.U. Cho, *The structure of regularity of near-rings*, Comm. Kor. Math. Soc. **2** (1987), 25–31.
3. Y.U. Cho, *Modified chain conditions for N-modules*, Comm Kor. Math. Soc. **5** (1990), 151–164.
4. K. Kuratowski, *Topology*, Vol. I, Academic Press, New York, 1966.
5. L. Li and B.M. Schein, *Strongly regular rings*, Semigroup Forum **32** (1985), 145–161.
6. K.D. Magill, *Semigroups and near-rings of continuous functions*, General Topology and its Relations to Modern Analysis and Algebra III, Proc. 3rd. Prague Top. Symp. 1971 (Academia, Prague, 1972), 283–288.
7. K.D. Magill, *Automorphism groups of laminated near-rings*, Proc. Edinburgh Math. Soc. **23** (1980), 97–102
8. K.D. Magill, P.R. Misra and U.B. Tewari, *Automorphism groups of laminated near-rings determined by complex polynomials*, Proc. Edinburgh Math. Soc. **26** (1983), 73–84.
9. K.D. Magill, P.R. Misra and U B. Tewari, *Finite automorphism groups of laminated near-rings*, Proc. Edinburgh Math. Soc. **26** (1983), 297–306.
10. G. Mason, *Strongly regular near-rings*, Proc. Edinburgh Math. Soc. **23** (1980), 27–36.
11. G. Pilz, *Near-rings*, North-Holland Pub. Co., Amsterdam, New York and Oxford, 1983.
12. S. Willard, *General topology*, Addition Wesley Pub Co., London and Ontario, 1970.

Department of Mathematics
Pusan Women's University
Pusan 616–060, Korea