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# ON THE BIFURCATION OF SUBHARMONIC ORBITS FOR GENERAL MAPS AT STRONG RESONANCES

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## 1. Introduction

This paper is concerned with the generalization of the results given by Kim and Lee ([12]) for a typical one-parameter family of area-preserving maps,so called Henon maps,to a *general* one-parameter family of maps at strong resonances.

The analysis for the bifurcation of the $n$-cycles at strong resonances (i.e., $n = 3, 4$) in this general case starts with imposing the assumptions that the complex conjugate eigenvalues of the linear part of a map lie on the unit circle in the complex plane and move *along the unit circle* as the parameter varies through zero.

To investigate the occurrence of subharmonic orbits from the origin for a general one-parameter family of maps, the theory of normal forms and the method of Liapunov-Schmidt reduction are also used here,but treated only briefly in Section 2 by referring the interested readers to the previous work ([12]) for more details.

The actual analysis and calculation in Section 3 and 4 employed to reveal the bifurcation pattern of the n-cycles ($n = 3,4$) in this general case yield many notable results,which,of course,should cover the previous results ([12]) obtained for a typical Henon map.

## 2. Preliminaries

Consider a general one-parameter family of maps on $\mathbf{R}^2$

$$F_\mu : \mathbf{R}^2 \longrightarrow \mathbf{R}^2 \tag{2.1}$$

where $F_\mu \in C^\infty$ and $\mu$ is a real parameter. We may assume that $F_\mu(0) = 0$ for any $\mu \in \mathbf{R}$.

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Let $D_x F_\mu(0) = A_\mu \in \mathbf{R}^{2\times 2}$ and $\lambda(\mu)$ , $\bar{\lambda}(\mu)$ be eigenvalues of $A_\mu$ for $\mu$ sufficiently small and let $\lambda_0 = \lambda(0)$ , $\bar{\lambda}_0 = \overline{\lambda(0)}$ be eigenvalues of $A_0$.

We assume that

$$(2.2) \qquad |\lambda(\mu)| = 1, \qquad \lambda_0 \neq \pm 1$$

$$(2.3) \qquad \frac{d}{d\mu} \arg \lambda(\mu)|_{\mu=0} > 0.$$

Notice that the condition (2.3) implies that the eigenvalues of the linear part of $F_\mu$ move along the unit circle as $\mu$ varies through 0 .

Since $F \in \mathcal{C}^\infty$, we can write

$$(2.4) \qquad \lambda(\mu) = \lambda_0(1 + \lambda_1 \mu + \mathcal{O}(|\mu|^2)).$$

From (2.2) and (2.3), we can write

$$(2.5) \qquad \begin{aligned} \lambda_1 &= 2\pi i a (a > 0), \\ \lambda_0 &= \exp(2\pi i \theta_0) \qquad (\theta_0 \neq 0, 1/2 \pmod 1) \end{aligned}$$

and

$$(2.6) \qquad \lambda(\mu) = \lambda_0 e^{2\pi i a \mu + \mathcal{O}(|\mu|^2)}.$$

By letting $z = x_1 + ix_2$,we can rewrite the given real map (2.1) in the following complex form

$$(2.7) \qquad z' = F_\mu(z) = \lambda(\mu) z + \sum_{l \geq 2} R_l(\mu, z, \bar{z}),$$

where

$$R_l(\mu, z, \bar{z}) = \sum_{p+q=l} c_{pq}(\mu) z^p \bar{z}^q, \; l \geq 2.$$

Now, we put (2.7) in a normal form by successive applications of a $\mu$-dependent change of variables of the following form

$$(2.8) \qquad z = w + \psi_l(\mu, w, \bar{w}) \equiv T_l(\mu, w), \quad l \geq 2,$$

where

$$\psi_l(\mu, w, \bar{w}) = \sum_{p+q=l} \gamma_{pq} w^p \bar{w}^q, \quad l \geq 2,$$

with a suitable choice of the coefficients $\gamma_{pq}(\mu)$. According to the theory of normal forms for maps ([4,6,7,8]), we can transform the given map (2.7) to the normal forms given in Kim and Lee ([12]) (Refer to Lemma 1 in [12]).

Now, following the method used in Kim ([12]), we can reduce the study of the occurrence of $n$-cycles into that of finding zeros of an algebraic function (so called bifurcation function) as stated in the following Lemma (For the proof, refer to the Lemma 2 in Kim ([12])).

LEMMA 1. *Assume that* $\lambda_0^n = 1 (n \geq 3)$ *and let* $x = (x_1, \cdots, x_n) \in \mathbf{C}^n$ *be a* $n$*-cycle of the map* $F_\mu$ *given in normal form. Let* $S$ *be a right-shift operator* $(x_1, \cdots, x_{n-1}, x_n) \to (x_2, \cdots, x_n, x_1)$ *and* $\mathcal{F}_\mu(x) = (F_\mu(x_1), \cdots, F_\mu(x_n))$. *Let* $y = Px, y = (y_1, \cdots, y_n) \in \mathbf{C}^n$, *where each column of* $P$ *consists of eigenvectors of* $S$. *Define a map* $\Phi$: $\mathbf{C}^n \times \mathbf{R} \longrightarrow \mathbf{C}^n$ *by* $\Phi(y, \mu) = P\mathcal{F}_\mu(P^{-1}y) - \Lambda y$, *where* $\Lambda =$ *diag* $(1, \bar{\lambda}_0, \cdots, \bar{\lambda}_0^{n-1})$. *Let* $L = D_y\Phi(0,0)$ *and write* $y = y_n v_n + w$, *where* $v_n = (0, \cdots, 0, 1) \in$ *Ker* $L$ *and* $w \in$ *Im* $L$. *Let* $E : \mathbf{C}^n \to$ *Im* $L$ *be a projection. Let* $z = \frac{1}{n} y_n$.

*Then finding the* $n$*-cycle* $(x_1, \cdots, x_n)$ *of* $F_\mu$ *is equivalent to solving the following equation in* $\mathbf{C}$:

$$\lambda_0 z = F_\mu(z) = \lambda(\mu) z + R(\mu, z, \bar{z}). \tag{2.9}$$

*Moreover, if we write*

$$x_1 \equiv \phi_\mu(z) \equiv z + \frac{1}{n} \sum_{j=1}^{n-1} w_j^*(nz, \mu), \tag{2.10}$$

*where* $w^* = (w_1^*, \cdots, w_{n-1}^*)$ *satisfies the equation*

$$E\Phi(y_n v_n + w^*(y_n, \mu)) \equiv 0$$

*then the other* $n$*-periodic points* $x_2, \cdots, x_n$ *are given by*

$$x_j = \phi_\mu(\lambda_0^{j-1} z) \; (j = 2, \cdots, n). \tag{2.11}$$

## 3. Bifurcation analysis of 3-cycles

**(i)** The case $n = 3$ and $c_{02}(0) \neq 0$.

In this case, $\lambda_0 = e^{2\pi i/3}$ and $F_\mu(z)$ has the normal form

$$F_\mu(z) = \lambda(\mu)z + c_{02}(\mu)\bar{z}^2 + \mathcal{O}(|z|^3) \tag{3.1}$$

where $\lambda(\mu) = \lambda_0(1 + \mu\lambda_1 + \mathcal{O}(|\mu|^2))$ with $\lambda_1 = 2\pi i a$ $(a > 0)$. Then (2.9) becomes

$$\mu\lambda_1 z + \bar{\lambda}_0 c_{02}(0)\bar{z}^2 + \mathcal{O}(|\mu|^2|z| + |\mu||z|^2 + |z|^3) = 0. \tag{3.2}$$

Letting $z = re^{2\pi i\phi}$ and separating the trivial solution $r = 0$,we have

$$2\pi i a\mu + \bar{\lambda}_0 c_{02}(0) r e^{-6\pi i\phi} + g(\mu, r, \phi) = 0, \tag{3.3}$$

where $g(\mu, r, \phi) = \mathcal{O}(|\mu|^2 + |\mu|r + r^2)$ and $g(\mu, r, \phi + 1/3) = g(\mu, r, \phi)$. Set

$$\begin{aligned} r &= 2\pi a \cdot \left|\frac{\mu}{c_{02}(0)}\right| \cdot (1 + r_1), \\ \phi &= \phi_0 + \phi_1, \\ \phi_0 &= -\frac{1}{36} - \frac{1}{6\pi}\arg\mu + \frac{1}{6\pi}\arg c_{02}(0) \pmod{1/3}. \end{aligned} \tag{3.4}$$

Substituting (3.4) in (3.3) and simplifying, we have

$$1 - e^{-6\pi i\phi_1}(1 + r_1) + g_2(\mu, r_1, \phi_1) = 0,$$

where

$$g_2(\mu, r_1, \phi_1) = (2\pi i a\mu)^{-1} g(\mu, 2\pi a\left|\frac{\mu}{c_{02}(0)}\right|(1 + r_1), \phi_0 + \phi_1) = \mathcal{O}(|\mu|).$$

Let

$$h(\mu, r_1, \phi_1) = 1 - e^{-6\pi i\phi_1}(1 + r_1) + g_2(\mu, r_1, \phi_1).$$

By the implicit function theorem, we have

$$r_1 = r_1(\mu), r_1(0) = 0, \phi_1 = \phi_1(\mu), \phi_1(0) = 0.$$

Consequently, we have from (3.4),

$$(3.5)\quad \begin{aligned} r &= 2\pi a \cdot \left|\frac{\mu}{c_{02}(0)}\right| \cdot (1+\mathcal{O}(|\mu|)) = 2\pi a\left|\frac{\mu}{c_{02}(0)}\right| + \mathcal{O}(|\mu|^2), \\ \phi &= -\frac{1}{36} - \frac{1}{6\pi}\arg\ \mu + \frac{1}{6\pi}\arg\ c_{02}(0) + \mathcal{O}(|\mu|) \quad (\text{mod } 1/3) \end{aligned}$$

and the coordinates of the 3-periodic points for the area-preserving map $F_\mu(z)$ in normal form are given, from (2.10), (2.11) and (3.5), by

$$(3.6)\quad \begin{aligned} x_1 &= \phi_\mu(z) \equiv z(\mu) + \mathcal{O}(|\mu||z| + |z|^2) \\ &= r(\mu)e^{2\pi i\phi(\mu)} + \mathcal{O}(|\mu|^2) = 2\pi a\left|\frac{\mu}{c_{02}(0)}\right| e^{2\pi i\phi_0} + \mathcal{O}(|\mu|^2), \\ x_2 &= \phi_\mu(\lambda_0 z), \\ x_3 &= \phi_\mu(\lambda_0^2 z). \end{aligned}$$

Note that as $\mu$ varies from $\mu < 0$ to $\mu > 0$,arg($\mu$) changes by $\pi$, and hence the orientation of the 3-cycle is reversed as $\mu$ crosses 0.

To examine the stability of the 3-cycle for the map

$$F_\mu(z) = \lambda(\mu)z + c_{02}(\mu)\bar{z}^2 + \mathcal{O}(|z|^3),$$

we consider the map

$$F_\mu^3(z) = (1 + 3\mu\lambda_1 + \mathcal{O}(|\mu|^2))z + 3\bar{\lambda}_0 c_{02}(0)\bar{z}^2 + \mathcal{O}(|\mu||z|^2 + |z|^3).$$

Then, we can easily see that the eigenvalues of the Jacobian $\partial(F_\mu^3(z), \bar{F}_\mu^3(z))/\ \partial(z,\bar{z})$ are real and hyperbolic and hence the 3-cycle is hyperbolic (saddle) on both sides of $\mu = 0$.

Thus, we have the following conclusion :

THEOREM 1. *Let $F_\mu : \mathbf{C} \longrightarrow \mathbf{C}$ be the general map given in* (2.7) *and assume that the conditions* (2.2) *and* (2.3) *hold and that* $\lambda_0^3 = 1$ $(\lambda_0 \neq \pm 1)$ *and* $c_{02} \neq 0$ *and $F_\mu(z)$ is put into the normal form* (3.1).

*Then a one-parameter family of 3-cycles* $\{(x_1(\mu), x_2(\mu), x_3(\mu)) \mid \mu \in \mathbf{R}\}$ *undergoes transcritical bifurcation from the origin and they are* *hyperbolic (saddle) on both sides of $\mu = 0$ and reverses the orientation as $\mu$ crosses 0. The 3-periodic points are given by* (3.6)

**(ii)** The case $n = 3$ and $c_{02}(0) = 0$.

In this case, we can remove the second order term because the coefficient $\gamma_{02}(\mu)$ of the transformation $z = w + \psi(\mu, w, \bar{w})$, $\psi(\mu, w, \bar{w}) = \sum_{p+q\geq 2} \gamma_{pq}(\mu) w^p \bar{w}^q$ becomes regular for $\mu$ near 0. Hence,after the change of variables, $F_\mu$ takes the form

$$F_\mu(z) = \lambda(\mu)z + \alpha(\mu)z^2\bar{z} + \beta(\mu)z^4 + \gamma(\mu)z\bar{z}^3 + \mathcal{O}(|z|^5), \tag{3.7}$$

where the coefficients $\alpha_0 \equiv \alpha(0), \beta_0 \equiv \beta(0)$ and $\gamma_0 \equiv \gamma(0)$ can be calculated from the original coefficients of $F_\mu(z)$, e.g.,

$$\alpha_0 \equiv c_{21}(0) + \frac{|c_{11}(0)|^2}{1 - \bar{\lambda}_0} + \frac{2\lambda_0 - 1}{\lambda_0(1 - \lambda_0)} c_{11}(0) c_{20}(0).$$

Here we assume that $\alpha_0, \beta_0, \gamma_0 \neq 0$. From (2.9), we have

$$\begin{aligned} &\mu\lambda_1 z + \bar{\lambda}_0\alpha_0 z^2\bar{z} + \bar{\lambda}_0\beta_0 z^4 + \bar{\lambda}_0\gamma_0 z\bar{z}^3 \\ &\qquad + \mathcal{O}(|\mu|^2|z| + |\mu||z|^3 + |\mu||z|^4 + |z|^5) = 0. \end{aligned} \tag{3.8}$$

Setting $z = re^{2\pi i\phi}$ and separating the trivial solution $r = 0$, we have

$$\begin{aligned} &2\pi i a\mu + \bar{\lambda}_0\alpha_0 r^2 + \bar{\lambda}_0\beta_0 r^3 e^{6\pi i\phi} + \bar{\lambda}_0\gamma_0 r^3 e^{-6\pi i\phi} \\ &\qquad + \mathcal{O}(|\mu|^2 + |\mu|r^2 + |\mu|r^3 + r^4) = 0. \end{aligned} \tag{3.9}$$

Set

$$\mu = \mu_0 r^2 + \mu_1 r^3, \qquad \phi = \phi_0 + \phi_1, \tag{3.10}$$

where $\mu_0$,$\mu_1$,$\phi_0$ and $\phi_1$ are to be determined. Substituting (3.10) into (3.9), we have

$$\begin{aligned} &(2\pi i a\mu_0 + \bar{\lambda}_0\alpha_0)r^2 \\ &\qquad + (2\pi i a\mu_1 + \bar{\lambda}_0(\beta_0 e^{6\pi i\phi} + \gamma_0 e^{-6\pi i\phi}))r^3 + \mathcal{O}(r^4) = 0. \end{aligned} \tag{3.11}$$

First, choose $\mu_0$ so that $2\pi i a\mu_0 + \bar{\lambda}_0\alpha_0 = 0$. Then

$$\mu_0 = \begin{cases} \frac{|\alpha_0|}{2\pi a}, & \text{if} \quad \arg\alpha_0 = \frac{\pi}{6} \pmod{2\pi} \\ \frac{-|\alpha_0|}{2\pi a}, & \text{if} \quad \arg\alpha_0 = \frac{7\pi}{6} \pmod{2\pi}. \end{cases} \tag{3.12}$$

Thus, note that if $\arg\alpha_0 \neq \pi/6 \pmod{\pi}$, there does not exist any 3-cycles bifurcating from the origin. Hence, from now on, we assume that

$$\arg\alpha_0 = \pi/6 \pmod{\pi}. \tag{3.13}$$

With this choice of $\mu_0$, (3.11) becomes

$$\bar{\lambda}_0(\beta_0 e^{6\pi i\phi} + \gamma_0 e^{-6\pi i\phi}) + 2\pi i a\mu_1 + \mathcal{O}(r) = 0. \tag{3.14}$$

In order to choose $\phi_0$ so that $\beta_0 e^{6\pi i\phi_0} + \gamma_0 e^{-6\pi i\phi_0} = 0$, we must have $|\beta_0| = |\gamma_0|$ and

$$\phi_0 = \frac{1}{12\pi}\arg(-\frac{\gamma_0}{\beta_0}) \pmod{1/6}. \tag{3.15}$$

If $|\beta_0| \neq |\gamma_0|$, there is no 3-cycle bifurcating from the origin. So, here, we also assume that

$$|\beta_0| = |\gamma_0| \neq 0. \tag{3.16}$$

Note that $\phi_0$ in (3.15) has two values

$$\begin{aligned} \phi_0^{(1)} &= \frac{1}{12\pi}\arg(-\frac{\gamma_0}{\beta_0}) \pmod{1/3}, \\ \phi_0^{(2)} &= \frac{1}{12\pi}\arg(-\frac{\gamma_0}{\beta_0}) + 1/6 \pmod{1/3}. \end{aligned} \tag{3.17}$$

Now from (3.14), we let

$$h(\mu_1, r, \phi) = \bar{\lambda}_0(\beta_0 e^{6\pi i\phi} + \gamma_0 e^{-6\pi i\phi}) + 2\pi i a\mu_1 + \mathcal{O}(r).$$

Then by the implicit function theorem, we know that

$$\mu_1 = \mu_1^{(j)}(r) = \mathcal{O}(r), \phi = \phi^{(j)}(r) = \phi_0^{(j)} + \mathcal{O}(r) \quad (j = 1, 2).$$

Thus, we have a pair of 3-cycles $z = re^{2\pi i\phi^{(j)}(r)}$ $(j = 1, 2)$ on one side of $\mu = 0$,where r is regarded as a parameter which is related to $\mu$ as

$$
\begin{aligned}
&\mu^{(1)} = \mu_0 r^2 + \mathcal{O}(r^4),\\
&\phi^{(1)} = \frac{1}{12\pi}\arg(-\frac{\gamma_0}{\beta_0}) + \mathcal{O}(r) \pmod{1/3},\\
&\mu^{(2)} = \mu_0 r^2 - \mathcal{O}(r^4),\\
&\phi^{(2)} = \frac{1}{12\pi}\arg(-\frac{\gamma_0}{\beta_0}) + 1/6 + \mathcal{O}(r) \pmod{1/3}.
\end{aligned}
\tag{3.18}
$$

Note that if $\arg\alpha_0 = \pi/6 \pmod{2\pi}$, we have a supercritical bifurcation and if $\arg\alpha_0 = 7\pi/6 \pmod{2\pi}$, subcritical bifurcation.

To study the stability of the pair of 3-cycles for the map

$$F_\mu(z) = \lambda(\mu)z + \alpha(\mu)z^2\bar{z} + \beta(\mu)z^4 + \gamma(\mu)z\bar{z}^3 + \mathcal{O}(|z|^5),$$

we consider the map

(3.19)
$$
\begin{aligned}
F_\mu^3(z) = (1 + 6\pi i a\mu)z &+ 3\bar{\lambda}_0\alpha_0 z^2\bar{z} + 3\bar{\lambda}_0\beta_0 z^4 + 3\bar{\lambda}_0\gamma_0 z\bar{z}^3\\
&+ \mathcal{O}(|\mu|^2|z| + |\mu||z|^3 + |\mu||z|^4 + |z|^5).
\end{aligned}
$$

Then we can easily check that one of the two 3-cycles on one side must be hyperbolic (saddle).

Therefore, we can state the following theorem.

THEOREM 2. *Let $F_\mu : \mathbf{C} \longrightarrow \mathbf{C}$ be the general map given in* (2.7) *and assume that the conditions* (2.2) *and* (2.3) *hold and that* $\lambda_0^3 = 1(\lambda_0 \neq \pm 1)$, $c_{02}(0) = 0$ *and* $F_\mu(z)$ *is put into a normal form* (3.7).

*Then, unless either* $\arg\alpha_0 = \frac{\pi}{6} \pmod{\pi}$ *or* $|\beta_0| = |\gamma_0|$ $(\neq 0)$, *there is no bifurcation of 3-cycles from the origin.*

*If both conditions hold,then a pair of one-parameter family of 3-cycles* $\{(x_1^{(j)}, x_2^{(j)}, x_3^{(j)}) \mid r \in \mathbf{R}^+, j = 1, 2\}$ *undergoes a supercritical (if* $\arg\alpha_0 = \frac{\pi}{6} \pmod{2\pi}$ *) or subcritical (if* $\arg\alpha_0 = \frac{7\pi}{6} \pmod{2\pi}$ *) bifurcation from the origin and the parameter r is related to* $\mu$ *as in* (3.18). *Moreover, on either side of* $\mu = 0$ *one of two families of 3-cycles is hyperbolic (saddle).*

## 4. Bifurcation analysis of 4-cycles

Let $\lambda_0 = e^{2\pi i/4} = i$. The normal form of $F_\mu(z)$ is

$$F_\mu(z) = \lambda(\mu)z + \alpha(\mu)z^2\bar{z} + \beta(\mu)\bar{z}^3 + \mathcal{O}(|z|^5). \tag{4.1}$$

where $\alpha(0) \equiv \alpha_0$ and $\beta(0) \equiv \beta_0$ can be computed from the coefficients of the original equation (See Kim ([12])) and we assume that $\alpha_0, \beta_0 \neq 0$. From (2.9), we have

$$\mu\lambda_1 z + \bar{\lambda}_0\alpha_0 z^2\bar{z} + \bar{\lambda}_0\beta_0\bar{z}^3 + g_1(\mu, z, \bar{z}) = 0, \tag{4.2}$$

where $g_1(\mu, z, \bar{z}) = \mathcal{O}(|\mu|^2|z| + |\mu||z|^3 + |z|^5)$. Setting $z = re^{2\pi i\phi}$ and separating the trivial solution $r = 0$, we have

$$2\pi a\mu - \alpha_0 r^2 - \beta_0 r^2 e^{-8\pi i\phi} + g(\mu, r, \phi) = 0, \tag{4.3}$$

where $g(\mu, r, \phi) = \mathcal{O}(|\mu|^2 + |\mu|r^2 + r^4)$.

To look for the principal part, put

$$\mu = \mu_0 r^2 + \mu_1 r^2, \qquad \phi = \phi_0 + \phi_1, \tag{4.4}$$

where $\mu_0, \mu_1, \phi_0$ and $\phi_1$ are to be determined. Substituting (4.4) in (4.3) and dividing by $r^2$, we have

$$(2\pi a\mu_0 - \alpha_0 - \beta_0 e^{-8\pi i\phi_0}) + 2\pi a\mu_1 + f_1(\mu, r, \phi) = 0, \tag{4.5}$$

where $f_1(\mu, r, \phi) = \mathcal{O}(r^2)$,$f_1(\mu, r, \phi + 1/4) = f_1(\mu, r, \phi)$. Choose $\mu_0$, $\phi_0$ so that

$$2\pi a\mu_0 - \alpha_0 - \beta_0 e^{-8\pi i\phi_0} = 0. \tag{4.6}$$

Then we must have

$$\begin{aligned} &|2\pi a\mu_0 - \alpha_0| = |\beta_0|, \\ &\phi_0 = -\frac{1}{8\pi}\arg\left(\frac{2\pi a\mu_0 - \alpha_0}{\beta_0}\right) \pmod{1/4}. \end{aligned} \tag{4.7}$$

Let $\mu_0^{(1)}, \mu_0^{(2)}$ be two solutions of $|2\pi a\mu_0 - \alpha_0| = |\beta_0|$. Then we have

$$\mu_0^{(1),(2)} = \frac{1}{2\pi a}\left\{\mathrm{Re}\ \alpha_0 \pm \sqrt{(|\beta_0|^2 - |\mathrm{Im}\ \alpha_0|^2)}\right\}. \tag{4.8}$$

Note that, since $\mu_0$ is real, we must have

$$\text{(4.9)} \qquad |\text{Im } \alpha_0| \leq |\beta_0|.$$

That is, if $|\text{Im } \alpha_0| > |\beta_0|$, there does not exist any 4-cycles bifurcating from the origin. Assume that (4.9) is satisfied. Once $\mu_0$ is determined from (4.8), then we know from (4.7) that $\phi_0$ has also two values

$$\text{(4.10)} \qquad \phi_0^{(j)} = -\frac{1}{8\pi} \arg\left(\frac{2\pi a \mu_0^{(j)} - \alpha_0}{\beta_0}\right) \; j = 1, 2. \quad (\text{mod } 1/4).$$

From (4.5), let

$$h(\mu_1, r, \phi) = (2\pi a \mu_0 - \alpha_0 - \beta_0 e^{-8\pi i \phi}) + 2\pi a \mu_1 + f_1(\mu, r, \phi).$$

Note that in order to apply implicit function theorem, we must have the strictly in equality sign in (4.9), i.e.,

$$\text{(4.11)} \qquad |\text{Im } \alpha_0| < |\beta_0|.$$

Then under the assumption (4.11), by the implicit function theorem, we know that

$$\mu_1 = \mu_1(r), \qquad \phi = \phi(r), \qquad \mu_1(0) = 0, \qquad \phi(0) = \phi_0$$

since $f_1(\mu, r, \phi)$ is an even function of r from the property in (4.5). We also know that $\mu(r), \phi(r)$ are even functions of $r$. Thus, we have

$$\text{(4.12)} \qquad \begin{aligned} \mu^{(j)}(r) &= \mu_0^{(j)} r^2 + \mathcal{O}(r^4) \quad (j = 1, 2) \\ \phi^{(j)}(r) &= \phi_0^{(j)} + \mathcal{O}(r^2) \end{aligned}$$

where $\mu_0^{(1),(2)}, \phi_0^{(1),(2)}$ are given in (4.8) and (4.10). From (4.8), we know that

$$\mu_0^{(1)} \cdot \mu_0^{(2)} = \frac{|\alpha_0|^2 - |\beta_0|^2}{4\pi^2 a^2}.$$

Hence we know that if $|\alpha_0| > |\beta_0|$, then $\mu_0^{(1)} \cdot \mu_0^{(2)} > 0$ and the so to families of 4-cycles bifurcate on the same side of $\mu = 0$. And if

$|\alpha_0| < |\beta_0|$, then $\mu_0^{(1)} \cdot \mu_0^{(2)} < 0$ and so they bifurcate on the opposite sides of $\mu = 0$ (i.e., transcritical bifurcation). If $|\alpha_0| = |\beta_0|$, then $\mu_0^{(1)} = 0, \mu_0^{(2)} = \frac{1}{\pi a}$ Re $\alpha_0 \neq 0$ and hence we know that

$$\mu^{(1)}(r) = \mathcal{O}(r^4),$$
$$\mu^{(2)}(r) = \mu_0^{(2)} r^2 + \mathcal{O}(r^4). \tag{4.13}$$

To study the stability of the 4-cycles for the map

$$F_\mu(z) = \lambda(\mu)z + \alpha(\mu)z^2\bar{z} + \beta(\mu)\bar{z}^3 + \mathcal{O}(|z|^5)$$

where $\alpha(0) \equiv \alpha_0, \beta(0) \equiv \beta_0$ are given by Kim ([12]), we consider the map

$$F_\mu^4(z) = [1 + 8\pi i a\mu + \mathcal{O}(|\mu|^2)]z - 4i\alpha_0 z^2\bar{z} - 4i\beta_0\bar{z}^3 + \mathcal{O}(|\mu||z|^3 + |z|^5).$$

If $\sigma_1, \sigma_2$ are the eigenvalues of the linear part of $F_\mu^4(z)$ at one of the 4 fixed points of one family, then we can easily see that if $|\alpha_0| < |\beta_0|$, then $\sigma_1$, $\sigma_2$ are real hyperbolic and if $|\alpha_0| > |\beta_0|$, then one of the families is hyperbolic. Hence we have the following conclusion :

THEOREM 3. *Let $F_\mu : \mathbf{C} \longrightarrow \mathbf{C}$ be the general map given in (2.7) and assume that the conditions (2.2) and (2.3) hold and that $\lambda_0^4 = 1(\lambda_0 \neq \pm 1)$ and $F_\mu(z)$ is put into the normal form (4.1), where $\alpha_0 \neq 0$, $\beta_0 \neq 0$.*

*Then if $|Im\ \alpha_0| > |\beta_0|$, there is no bifurcation of 4-cycles from the origin. If $|Im\ \alpha_0| < |\beta_0|$, then we have two one-parameter families of 4-cycles $\{(x_1^{(j)}(r), x_2^{(j)}(r), x_3^{(j)}(r), x_4^{(j)}(r)) \mid r \in \mathbf{R}^+, j = 1,2\}$ bifurcating from the origin and those 4-cycles are given by $x_k^{(j)} = x_k^{(j)}(r) = re^{2\pi i(\phi_0^{(j)} + \frac{k-1}{4})} + \mathcal{O}(r^3)$, $(j = 1,2, k = 1,2,3,4)$ where the parameter r is related to $\mu$ as in (4.4), (4.8) and (4.10).*

*Moreover, if $|\alpha_0| > |\beta_0|$, then the two families bifurcates on the same sides of $\mu = 0$ and one of the families is hyperbolic (saddle) and if $|\alpha_0| < |\beta_0|$,the two families bifurcate on the opposite side of $\mu = 0$ and both are hyperbolic.*

## References

1. R. Mackay, *Renormalization in area-preserving maps*, Ph.D. thesis, Princeton Univ. (1982).
2. J. Van Der Weele, H. Capel, T. Valking and T. Post, *The squeeze effect in non-intergrable Hamiltonian systems*, Phys. **A147** (1988), 499–532.
3. V.I. Arnold, *Ordinary Differential Equations*, 1981.
4. V.I. Arnold, *Geometrical Methods in the Theory of Ordinary Differential Equation*, Springer-Verlag, New York, 1983
5. G. Ioose and D.D. Joseph, *Elementary Stability and Bifurcation Theory*, Springer-Verlag, New York, Heidelberg, Berlin, 1980.
6. G. Ioose, *Bifurcation of Maps and Applications*, North- Holland, 1979.
7. A. Cushman, A. Deprit and R. Mosak, *Normal forms and representation theory*, J. Math. Phys. **24** (1983), 2102–2117.
8. J. Guckenheimer and P Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*, Springer-Verlag, New York, 1983
9. S. Chow and J Hale, *Methods of Bifurcation Theory*, Springer-Verlag, New York, 1982.
10. M. Golubitsky, I. Stewart and G. Schaeffer, *Singularities and Groops in Bifurcation Theory, Vol. I, II*, Springer-Verlag, New York, 1988.
11. J. Hale, *Ordinary Differential Equations*, 1980.
12. Y.I Kimand E.K. Lee, *J. Kor. Math Soc.*

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