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SOME PROPERTIES OF SUBSPACES OF THE SPACE OF FOURIER ULTRAHYPERFUNCTIONS AND RELATED PROBLEMS

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0. Introduction

This study singles out two families of subspaces of Fourier ultrahyperfunctions and three distributive lattices $\mathcal{W}$, $\mathcal{S}$ and $\mathcal{M}$ and discusses related problems. It defines Banach space $\Pi(K,m)$ and considers inductive limits and projective limits of the Banach spaces and their duals. We comment on the Fourier transform and the Laplace transform of some functionals. We consider direct sums of subspaces. Finally, we study the lattice of subspaces of Fourier ultrahyperfunctions.

1. Auxiliary distributive lattices and inequalities

Let $\mathcal{W}$ be the set of all nega-support functions of sets $\mathcal{M} \subset R^n$ of the family $\mathcal{K} = \{M : M = \overline{\cup M_a},\ M_a \subset R^n \text{ convex closed},\ p \in M_a\}$: $\mathcal{W} = \{W_M : M \in \mathcal{K}\}$, where p is a fixed point in R^n. This set $\mathcal{W}$ is partially ordered by the relation $\leq$. The operations of intersection and union are defined on $\mathcal{W}$ by the relations.

$$W_M \cap W_N = W_{M\cup N}, \quad \text{and} \quad W_M \cup W_N = W_{M\cap N}.$$

Then $\mathcal{W}$ forms a distributive lattice. Similary, we let $\mathcal{S} = \{S_M : M \in \mathcal{K}\}$, where S_M is a support function of $\mathcal{M}$. We define the operations of intersection and union on S as follows:

$$S_M \cap S_N = S_{M\cap N} \quad \text{and} \quad S_M \cup S_N = S_{M\cup N}.$$

Then $\mathcal{S}$ forms a distributive lattice.

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We write $W_M \lessdot W_N$ if $W_M(x) < W_N(x)$ for all $x \neq 0$. Let $\mathcal{M} = \{m = (m_1, \cdots, m_n) : m_i(i = 1, \ldots, n)$ continuous homogeneous functions on R and $m(x) = \sum_{i=1}^n m_i(x_i)$ for $x = (x_1, \cdots, x_n) \in R^n\}$. We note that m is clearly a continuous homogeneous function on R^n.

We define, for $m,\ m' \in \mathcal{M}$, $m = (m_1, \cdots, m_n)$, $m' = (m'_1, \cdots, m'_n)$,

$$m \wedge m' = (m_1 \wedge m'_1, \cdots, m_n \wedge m'_n), \quad m \vee m' = (m_1 \vee m'_1, \cdots, m_n \vee m'_n).$$

Then $\mathcal{M}$ is a distributive lattice.

Theorem 1.1. *For each function $W_M \in \mathcal{W}$, there exists a sequence $\{m_\nu\} \subset \{m \in \mathcal{M} : W_M \lessdot m\}$ with the following properties:*
(a) $m_1 \gg m_2 \gg \cdots,$
(b) $\bigcap_{1 \le \nu < \infty} m_\nu = W_M,$
(c) *For any function $S_N \in \mathcal{S}$ with $S_N \gg W_M$ there is an index μ such that $m_\mu \lessdot S_N$.*

Proof. We define $m_\nu(\nu = 1, 2, \cdots)$ by the equality

$$\begin{aligned} m_\nu(\sigma) &= \frac{1}{n}\max\{W_M(s) - \nu|s_1 - \sigma_1| + \frac{1}{\nu} : s = (s_1, \cdots, s_n) \in S^{n-1}\} \\ &\quad + \cdots + \frac{1}{n}\max\{W_M(s) - \nu|s_n - \sigma_n| + \frac{1}{\nu} : s \in S^{n-1}\}, \\ &\quad \sigma = (\sigma_1, \cdots, \sigma_n) \in S^{n-1} \end{aligned}$$

and extend it to the whole of R^n by homogeneity.

We denote by

$$m_{\nu_i}(\sigma_i) = \frac{1}{n}\max\{W_M(s) - \nu|s_i - \sigma_i| + \frac{1}{\nu} : s = (s_1, \cdots, s_n) \in S^{n-1}\}.$$

Then $m_\nu(\lambda\sigma) = \lambda m_\nu(\sigma) = \lambda \sum_{i=1}^n m_{\nu_i}(\sigma_i)$ for every $\lambda > 0$, $\sigma \in S^{n-1}$. Clearly m_{ν_i} is a continuous homogeneous function on R. Hence $m_\nu \in \{m \in \mathcal{M} : W_M \lessdot m\}$ and we have (a) and (b). We can show (c) as similar way of 9.2 Chapter III([2]).

Theorem 1.2. *For any function $W_M \in \mathcal{W}$, $\sigma \in S^{n-1}$ and $t \in R$ such that $W_M(\sigma) < t$, there exists a convex compact set $K \subset R^n$ such that $W_M \lessdot S_K$ and $S_K(\sigma) < t$.*

Proof. We can prove it as similar way of 9.3 Chapter III([2]).

THEOREM 1.3. *For any function $W_M \in \mathcal{W}$ there exists a sequence of compact sets $K_\nu \in \{K \subset R^n$ convex compact : $W_M \lessdot S_K\}$ with the following properties:*

For every function $S_K \in \mathcal{S}$ with $W_M \lessdot S_K$ there is an index N such that $\cap_{1\le\nu\le N} S_{K_\nu} \lessdot S_K$. In particular, $\cap_{\nu=1}^{\infty} S_K = W_M$.

Proof. We can prove it as similar way of 9.4 Chapter III([2]).

2. The Banach space $\Pi(K,m)$, inductive limits and projective limits

Let $K \subset R^n$ be convex compact with nonempty interior and let $m \in M$. By $\Pi(K,m)$ we denote the Banach space of all functions ϕ which are continuous on $T(K)$ and holomorphic in $T(\overset{\circ}{K})$ and have the finite norm:

$$\|\phi\|_{K,m} = \sup\{\exp(m(x))\phi(z) : z = x + iy \in T(K)\}.$$

The family $\{\Pi(L,m) : K \Subset L, W_M \lessdot m\}$ forms a compact inductive spectrum of Banach spaces. The inductive limit

$$\overrightarrow{\Pi}(K,M) = \operatorname*{ind\,lim}_{K \Subset L,\ W_M \lessdot m} \Pi(L,m) \tag{2.1}$$

of this spectrum is a DFS-space. Similary, let $U \subset R^n$ be open convex and $S_M \in \mathcal{S}$. The family $\{\Pi(K,m) : K \Subset U, m \lessdot S_M\}$ forms a compact projective spectrum of Banach spaces. Hence we can define the space

$$\overleftarrow{\Pi}(U,M) = \text{proj}\lim\nolimits_{K \Subset U,\ m \lessdot S_M} \Pi(K,m) \tag{2.2}$$

$\overleftarrow{\Pi}(U,M)$ is an FS-space. We define

$$\begin{aligned} \overrightarrow{\Pi}(\{0\},\{0\}) &= \operatorname*{ind\,lim}_{\{0\} \Subset K,\ W_{\{0\}} \lessdot m} \Pi(K,m), \\ \overleftarrow{\Pi}(R^n,R^n) &= \operatorname*{ind\,proj}_{K \Subset R^n,\ m \lessdot S_{R^n}} \Pi(K,m). \end{aligned} \tag{2.3}$$

We note that the extreme spaces are same consequences of our earlier results (cf. [3],[5] and [6]).

The dual spaces $\overrightarrow{\Pi}'(\{0\},\{0\})$ of $\overrightarrow{\Pi}(\{0\},\{0\})$ is the space of Fourier hyperfunctions (cf.[3]). The dual space $\overleftarrow{\Pi}'(R^n,R^n)$ of $\overleftarrow{\Pi}(R^n,R^n)$ is the space of Fourier ultrahyperfunctions, which is the largest space in the case of spaces we shall consider in the sequal. We have the continuous dense imbeddings

$$\overleftarrow{\Pi}(R^n,R^n) \subset \overrightarrow{\Pi}(K',M') \subset \overrightarrow{\Pi}(K'',M'') \subset \overrightarrow{\Pi}(K,R^n)$$

for all $K' \supset K'' \supset K$ and $M' \subset M'' \subset R^n$. Hence the inverse chain of imbeddings is valid for the dual spaces. In particular, $\overrightarrow{\Pi}'(K,M)$ is an LCS-subspace of $\overleftarrow{\Pi}'(R^n,R^n)$ for any convex compact set K and any function $W_M \in \mathcal{W}$. Similary, there are continuous imbeddings

$$\overleftarrow{\Pi}(R^n,R^n) \subset \overleftarrow{\Pi}(U',M') \subset \overleftarrow{\Pi}(U'',M'') \subset \mathcal{H}(T(U))$$

for all $U' \supset U'' \supset U$ and $M' \supset M''$. Moreover, the first imbedding (hence also the second) is dense. In particular, $\overleftarrow{\Pi}'(U,M)$ is an LCS-subspace of $\overleftarrow{\Pi}(R^n,R^n)$ for each open convex set U and each function $S_M \in \mathcal{S}$.

With the help of Theorem 1.1(c), for any convex compact set K, and convex open set U and any functions $W_M \in \mathcal{W}$ and $S_K \in \mathcal{S}$ such that $K \subset U$ and $W_M < S_N$, the continuous, dense imbedding $\overleftarrow{\Pi}(U,N) \subset \overrightarrow{\Pi}(K,M)$ is valid along with the dual imbedding for the dual spaces. Using these relations, we can obtain the equalities

$$(2.4)\qquad \begin{aligned} \overleftarrow{\Pi}(U,N) &= \operatorname*{ind\,proj}_{K \Subset U,\ W_M \lessdot S_N} \overrightarrow{\Pi}(K,M), \\ \overrightarrow{\Pi}(K,M) &= \operatorname*{ind\,lim}_{K \Subset U,\ W_M \lessdot S_N} \overleftarrow{\Pi}(U,N) \end{aligned}$$

connecting the spaces $\overleftarrow{\Pi}(U,N)$ and $\overrightarrow{\Pi}(K,M)$, along with the dual spaces

$$\overleftarrow{\Pi}'(U,N) = \operatorname*{ind\,lim}_{K\subset U,\ W_M \lessdot S_N} \overrightarrow{\Pi}'(K,M),$$
(2.5)
$$\overrightarrow{\Pi}'(K,M) = \operatorname*{ind\,proj}_{K\subset U,\ W_M \lessdot S_N} \overleftarrow{\Pi}'(U,N)$$

3. The Fourier transform and the Laplace transform

Let $K \subset R^n$ and $K' \subset R^n$ be convex compact sets with nonempty interior. Let us define $\Pi(K, S_{K'})$ be the space $Q_S(T(K); K')$ and $\Lambda(K, W_{K'})$ be the space $Q_W(T(K); K')$ (see [5]). We know that $K \Subset K'$ iff $W_{K'} \lessdot W_K$. By Theorem 1.1, there exists a sequence $m_\nu \in \mathcal{M}$ such that $m_1 \gtrdot m_2 \gtrdot \cdots m_\nu \gtrdot \cdots \gtrdot W_M$ and $\lim_{\nu\to\infty} m_\nu = W_K$ and also similarly there exists a sequence $m_\nu \in \mathcal{M}$ such that $m_1 \lessdot m_2 \lessdot \cdots \lessdot m_\nu \lessdot \cdots \lessdot S_K$ and $\lim_{\nu\to\infty} m_\nu = S_K$.

Let $\phi \in \Pi(K, S_{K'})$. The Fourier transform $\mathcal{F}[\phi]$ is defined by

$$\mathcal{F}[\phi](z) = \int \exp(iz\zeta)\phi(\zeta)d^n\xi, \qquad \zeta = \xi + i\eta, \quad \eta \in K.$$

Then, we can represent the results of Park ([5]) in case of spaces Π, $\overleftarrow{\Pi}$ and $\overrightarrow{\Pi}$ and their dual spaces.

For a convex compact set $K \subset R^n$ with nonempty interior and $m \in M$, we define $\Lambda(K,m) = \Pi(K,-m)$. We define

$$\overrightarrow{\Lambda}(K,U) = \operatorname*{ind\,lim}_{L\Supset K\ m\lessdot S_U} \Lambda(L,m),$$

$$\overleftarrow{\Lambda}(U,K) = \operatorname*{ind\,proj}_{K\subset U\ W_K\lessdot m} \Lambda(K,m).$$

We define the Laplce transform $\mathcal{L}[g]$ of a functional $g \in \Lambda'(K, W_{K'})$ (resp. $g \in \overrightarrow{\Lambda}'(K,U)$, or $g \in \overleftarrow{\Lambda}'(U,K)$) by the equation.

$$\mathcal{L}[g](z) = (g, \exp(iz)) \equiv (g(\zeta)), \exp(iz\zeta))$$

for all $z \in T(K')$ (resp. $z \in T(U)$, or $z \in T(K)$).

THEOREM 2.1. *Let $K \Subset L$ and $L' \Subset K'$. Then the Laplace transformation*

$$\mathcal{L} : \Lambda'(K, W_{K'}) \longrightarrow \Lambda(L', W_L)$$

is a continuous linear mapping.

THEOREM 2.2. *The Laplace transformation $\mathcal{L}$ determines topological isomorphisms:*

$$\mathcal{L} : \overrightarrow{\Lambda}'(K, U) \cong \overleftarrow{\Lambda}(U, K)$$
$$\mathcal{L} : \overleftarrow{\Lambda}'(U, K) \cong \overrightarrow{\Lambda}(K, U).$$

4. Direct sums of subspaces

The following conditions will be assumed to hold:
(a) X and Y are Hausdorff reflexive, locally convex spaces and their duals X' and Y' are also Hausdorff.
(b) ε is a family of FS-spaces, and ε' is the family of the dual spaces.
(c) $X \subset E \subset Y$, where both inclusions are dense and continuous for all $E \in \varepsilon$.

Assuming always that $E_i \in \varepsilon$ $(i = 1, \ldots, n)$, we set

$$N = \{(\phi, \cdots, \phi) : \phi \in \cap_{1 \leq i \leq n} E_i\}$$

and

$$L = \{(g, \cdots, g) : g \in \cap_{1 \leq i \in n} E_i'\}.$$

The space N (resp. L) is a closed subspace of XE_i (resp. XE_i') and N (resp. L) is an FS-space (resp. a DFS-space).

PROPOSITION 3.1.

$$\cap_{1 \leq i \leq n} E_i \cong N \quad \text{and} \quad \cap_{1 \leq i \leq n} E_i' \cong L.$$

In particular, $\cap_{1 \leq i \leq n} E_i$ is an FS-space and $\cap_{1 \leq i \leq n} E_i'$ is a DFS-space.

PROPOSITION 3.2. (Prop. 28 ([4])) *Let E be the inductive limit of the convex spaces E_γ by the mappings u_γ. We define a mapping u of σE_γ into E by putting $u(\Sigma X_\gamma) = \Sigma u_\gamma(X_\gamma)$. Then E is isomorphic to the quotient space $\Sigma E_\gamma / u^{-1}(0)$.*

COROLLARY 3.3. *Set $\check{N} = \{(\phi_1, \cdots, \phi_n) : \phi_i \in E_i, \sum_1^n \phi_i = 0\}$ and $\check{L} = \{(g_1, \cdots, g_n) : g_i \in E_i', \sum_1^n g_i = 0\}$. Then $E_1 \cup \cdots \cup E_n \cong \sum_1^n E_i/\check{N}$ and $E_1' \cup E_2' \cup \cdots \cup E_n' \cong \sum_1^n E_i'/\check{L}$.*

5. The lattice of subspaces of Fourier ultrahyperfunctions

Let $\vec{\pi}(K) = \{\vec{\Pi}(K, M) : W_M \in \mathcal{W}\}$ be a partially ordered (by a relation $\subset$) set, and let $X \subset \mathcal{W}$.

If K is a convex compact set in R^n with nonempty interior and χ is a family in $\mathcal{W}$, then

$$\vee_{W_M \in \chi} \vec{\Pi}(K, M) = \vec{\Pi}(K, \cup_{W_M \in \chi} M), \tag{5.1}$$

and if the family χ is finite, then also

$$\wedge_{W_M \in \chi} \vec{\Pi}(K, M) = \vec{\Pi}(K, \cap_{W_M \in \chi} M). \tag{5.2}$$

In both cases the dual equalities are valid for the dual spaces, with the meet and join taken in $\overleftarrow{\Pi}'(R^n, R^n)$.

$$(\vee_{W_M \in \chi} \vec{\Pi}(K, M))' = \wedge_{W_M \in \chi} \vec{\Pi}'(K, M) = \vec{\Pi}'(K, \cup_{W_M \in \chi} M), \tag{5.3}$$

$$(\wedge_{W_M \in \chi} \vec{\Pi}(K, M))' = \vee_{W_M \in \chi} \vec{\Pi}'(K, M) = \vec{\Pi}'(K, \cap_{W_M \in \chi} M), \tag{5.4}$$

Hence we have the followings:

PROPOSITION 5.1. *The family $\vec{\pi}(K) = \{\vec{\Pi}(K, M) : W_M \in \mathcal{W}\}$ (resp. $\vec{\pi}'(K) = \{\vec{\Pi}'(K, M) : W_M \in \mathcal{W}\}$) forms a distributive lattice* *of subspaces which is anti-isomorphic (resp. isomorphic) to the lattice $\mathcal{W}$ of nega-support functions.*

Analogous assertions are valid also for the space $\overleftarrow{\Pi}(U,M)$. If U is a convex open set in R^n and χ is a family in $\mathcal{S}$, then

$$(5.5)\qquad \wedge_{S_M\in\chi}\overleftarrow{\Pi}(U,M)=\overleftarrow{\Pi}(U,\cup_{S_M\in\chi}M),$$

and if χ is finite, then also

$$(5.6)\qquad \vee_{S_M\in\chi}\overleftarrow{\Pi}(U,M)=\overleftarrow{\Pi}(U,\cap_{S_M\in\chi}M),$$

In the both cases the dual equalities are valid for the dual spaces, with the meet and join taken in $\overleftarrow{\Pi}'(R^n,R^n)$.

$$(5.7)\qquad (\wedge\overleftarrow{\Pi}(U,M))'=\vee\overleftarrow{\Pi}'(U,M)=\overleftarrow{\Pi}'(U,\cup M)$$

and if χ is finite, then also

$$(5.8)\qquad (\vee\overleftarrow{\Pi}(U,M))'=\wedge\overleftarrow{\Pi}'(U,\cap M).$$

As a consequence we have the following:

PROPOSITION 5.2. *Let U be a convex open set in R^n. The family $\overleftarrow{\pi}(U)=\{\overleftarrow{\Pi}(U,M):S_M\in\mathcal{S}\}$ (resp. $\overleftarrow{\pi}'(U)=\{\overleftarrow{\Pi}'(U,M):S_M\in\mathcal{S}\}$) is a distributive lattice of subspaces which is anti-isomorphic (resp. isomorphic) to the lattice $\mathcal{S}$ of support functions.*

The Proposition 5.1 implies, in particular, that for any finite family $\chi=\{W_{M_1},\cdots,W_{M_n}\}\subset\mathcal{W}$ we have the equality

$$(5.9)\qquad \cap_{1\le\nu\le n}\overrightarrow{\Pi}'(K,M_\nu)=\overrightarrow{\Pi}'(K,\cup_{\nu=1}^n M_\nu).$$

It turns out that a similar equality also holds for an arbitrary (infinite family).

THEOREM 5.3. *Let K be a convex compact set in R^n with non-empty interior and let χ be a family of functions from $\mathcal{W}$. Set $M_0=\overline{\cup_{W_M\in\chi}M}$. Then $\cap_{W_M\in\chi}\overrightarrow{\Pi}'(K,M)=\overrightarrow{\Pi}'(K,M_0)$.*

REMARK. If we define $W_{M_0}(\xi) = \inf\{W_M(\xi) : W_M \in \chi\}$ then $W_{M_0} = W_{\overline{\cup M}}$ where union is taken for $W_M \in \chi$.

Proof. By definition $W_{M_0} \in \mathcal{W}$. Set $\cap\overrightarrow{\Pi}' = \cap_{W_M \in \chi}\overrightarrow{\Pi}'(K, M)$. Then by Section 2 we have continuous inclusions $\overrightarrow{\Pi}'(K, M_0) \subset \cap\overrightarrow{\Pi}'$ (by definition, $W_{M_0} \leq W_M$ for all $W_M \in \chi$). Thus, in order to prove the theorem, it suffices to show the existence of a continuous inverse inclusion $\cap\overrightarrow{\Pi}' \subset \overrightarrow{\Pi}'(K, M_0)$. Let us fix $L \Supset K$ and $W_M \gg W_{M_0}$. There exist functions $W_{M_1}, \cdots, W_{M_n} \in \chi$ such that $\cap_{i=1}^n W_{M_i} = W_{\cup_{i=1}^n M_i} \lessdot W_M$. Furthermore, by the definition of the projective limit,

$$\cap\overrightarrow{\Pi}' \subset \cap_{\nu=1}^n \overrightarrow{\Pi}'(K, M_\nu) = \overrightarrow{\Pi}'(K, \cup_{\nu=1}^n M_\nu)$$

and finally

$$\overrightarrow{\Pi}'(K, \cup_{\nu=1}^n M_\nu) \subset \overrightarrow{\Pi}'(L, W_M),$$

where the inclusion is continuous because $L \Supset K$ and $W_M \gg W_{\cup_1^n M_\nu}$.

Therefore, $\cap\overrightarrow{\Pi}'$ imbeds continuously into $\Pi'(L, W_M)$ for all $L \Supset K$ and $W_M \gg W_{M_0}$. According to the definition of the projective limit this means the existence of the desired inclusion. By a slight modification of the argument of Theorem 5.3 we arrive at the following conclusion:

THEOREM 5.4. *Let U be a convex open set in R^n and Ψ be a family of support functions from $\mathcal{S}$. Set $M_0 = \overline{\cup M}$, where union is taken for all $S_M \in \Psi$. Then $S_{M_0} = \vee_{S_M \in \Psi} S_M$, $\cap_{S_M \in \Psi}\overleftarrow{\Pi}(U, M) = \overleftarrow{\Pi}(U, M_0)$ and hence $\cup_{S_M \in \Psi}\overleftarrow{\Pi}'(U, M) = \overleftarrow{\Pi}'(U, M_0)$.*

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