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MAGNETOHYDRODYNAMIC WAVE PROPAGATION IN THE "IONOSPHERE" OF THE CENTRAL BLACK HOLE IN AN ACTIVE GALACTIC NUCLEUS

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ABSTRACT

An axisymmetric, stationary electrodynamic model of the central engine of an active galactic nucleus has been well formulated by Macdonald and Thorne. In this model the relativistic region around the central black hole must be filled by highly conducting plasma and the equations of magnetohydrodynamics are then satisfied. In this paper we analyze magnetohydrodynamic wave propagation in this region. We find that there are three distinct types of waves — the Alfvén wave and two magnetosonic waves. The wave equations turn out to be not very different from those in nonrelativistic case except they are redshifted.

I. INTRODUCTION

Most astrophysicists believe that central engines in active galactic nuclei (hereafter AGNs) are related to supermassive black holes (e.g., see Begelman et al. 1984). An axisymmetric, stationary electrodynamic model was well formulated in the language of the "3+1"-spacetime formalism by Macdonald and Thorne (1982, hereafter MT), which consists of the supermassive black hole surrounded by a magnetized accretion disk. The magnetosphere of the Macdonald and Thorne (hereafter M-T) model can be divided into three regions ($\mathbf{E}$, $\mathbf{B}$, ρ_e, and $\mathbf{j}$ have their usual definitions in electrodynamics):

1) *degenerate region*; the stretched event horizon of the black hole and the surface of the magnetized accretion disk are degenerate,

$$\mathbf{E} \cdot \mathbf{B} = 0. \tag{1.1a}$$

2) *force-free region*; the zones closest to the black hole are force-free,

$$\mathbf{E} \cdot \mathbf{B} = 0 \quad \text{and} \quad \rho_e \mathbf{E} + \mathbf{j} \times \mathbf{B} = 0. \tag{1.1b}$$

3) *acceleration region*; the zones farther from the black hole are called the acceleration region.

These regions must be filled by highly conducting plasma and in the force-free region, in particular, the equations of magnetohydrodynamics (hereafter MHD) will be well satisfied because condition (1.1b) guarantees infinite conductivity of plasma. In this paper we will call this region the "ionosphere" of the black hole. Concentrating on low frequency phenomena, we will investigate MHD wave propagation in the ionosphere. In §II we will summarize the electromagnetic fields in the ionosphere. In §III we will derive fundamental equations and consider astrophysical implications. Throughout this paper $(-+++)$ signs will be used with units such that $c = G \equiv 1$. Greek indices will run 0 to 3 while Latin indices 1 to 3.

II. MAGNETOSPHERE OF THE MACDONALD-THORNE MODEL

In black hole astrophysics it is useful to employ a fiducial observer (hereafter FIDO) who measures physical quantities in his neighborhood. For the full analysis readers are directed to Thorne, Price and Macdonald (1986, hereafter TPM). Each FIDO is at rest with respect to the black hole and he never moves from his fixed location. Throughout this paper we define all the electrodynamic quantities at a point in the ionosphere as those measured by the FIDO at that point, using his own proper time τ. Since τ is not a global coordinate time, we also use the universal time t to define a slicing of spacetime. They are related to each other by the lapse function α ,

$$\alpha \equiv \frac{d\tau}{dt}. \tag{2.1}$$

The M-T model is axisymmetric and stationary. Therefore, it is based on two assumptions such as (MT, eq. [4.1]),

$$\mathbf{m} \cdot \nabla f = 0, \quad \mathcal{L}_{\mathbf{m}} \mathbf{f} = \mathbf{0} \tag{2.2a}$$

and

$$\frac{\partial f}{\partial t} \equiv \dot{f} = 0, \quad \frac{\partial \mathbf{f}}{\partial t} \equiv \dot{\mathbf{f}} = \mathbf{0}, \tag{2.2b}$$

where $\mathcal{L}$ means the Lie derivative, $\mathbf{m}$ is a Killing vector of the axisymmetry, and f, $\mathbf{f}$ are any scalar and vector, respectively.

In this model the total electric current passing downward through an $\mathbf{m}$-loop, $I(\mathbf{x})$, and total magnetic flux passing upward through an $\mathbf{m}$-loop, $\Psi(\mathbf{x})$, are defined by (MT, eq. [4.2]),

$$I(\mathbf{x}) \equiv -\int_{\Sigma} \alpha \mathbf{j} \cdot d\mathbf{\Sigma} \tag{2.3a}$$

and

$$\Psi(\mathbf{x}) \equiv \int_{\Sigma} \mathbf{B} \cdot d\mathbf{\Sigma}, \tag{2.3b}$$

where Σ is any surface surrounded by an $\mathbf{m}$-loop, but not intersecting the black hole, and $d\Sigma$ is the infinitesimal normal vector of the surface.

In terms of I and Ψ the poloidal and toroidal components of the electric and magnetic fields seen by FIDOs in the ionosphere are (MT, eqs. [4.4], [4.5], [4.8], and [4.9])

$$\mathbf{E}^T = 0, \tag{2.4a}$$

$$\mathbf{E}^P = \mathbf{E}, \tag{2.4b}$$

$$\mathbf{B}^T = -\frac{2I}{\alpha\tilde{\omega}}\mathbf{e}_{\hat{\varphi}}, \tag{2.4c}$$

and

$$\mathbf{B}^P = -\frac{\mathbf{e}_{\hat{\varphi}} \times \nabla\Psi}{2\pi\tilde{\omega}}, \tag{2.4d}$$

where $\tilde{\omega}$ is the magnitude of $\mathbf{m}$ and $\mathbf{e}_{\hat{\varphi}}$ is the unit toroidal vector (see eq. [3.4c]).

III. MAGNETOHYDRODYNAMIC WAVE EQUATIONS

1. Maxwell Equations in the Ionosphere

In this paper we assume that the hole is so slowly-rotating that we may use the Schwarzschild metric as a good approximation. Since the rotational energy of the black hole in the M-T model is being extracted by the Blandford-Znajek process (Blandford and Znajek 1977), this assumption may fit well at the end of the evolution of galactic nuclei.

From the point of view of the 3+1-formalism, spacetime is a foliation of spacelike hyperspaces connected by timelike curves. If n^μ is the orthonormal timelike vector of the time slice, then $\gamma_{\mu\nu}$, the intrinsic curvature of the hyperspace, is defined by

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_{\mu\nu}, \tag{3.1}$$

where $g_{\mu\nu}$ means the usual 4-dimensional spacetime metric tensor. Now $g_{\mu\nu}$ is set by (Arnowitt, Deser, and Misner 1962)

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_k\beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix}, \tag{3.2}$$

where β^i the shift vector with $\beta_i = \gamma_{\mu\nu}\beta^j$. According to our assumption, nonvanishing coefficients of the metric in the spherical coordinates (r, θ, φ) are close to those of the Schwarzschild metric,

$$\alpha \simeq \left(1 - \frac{2M}{r}\right)^{1/2}, \tag{3.3a}$$

$$\gamma_{rr} \simeq \left(1 - \frac{2M}{r}\right)^{-1/2}, \tag{3.3b}$$

$$\gamma_{\theta\theta} \simeq r^2, \tag{3.3c}$$

and

$$\gamma_{\varphi\varphi} \simeq r^2 \sin^2\theta. \tag{3.3d}$$

Notice also that $\tilde{\omega}$, the magnitude of the Killing vector $\mathbf{m}$, now becomes equal to $r\sin\theta$. In terms of these the FIDO frame in absolute space will be (TPM, eq. [2.3a])

$$\mathbf{e}_{\hat{r}} \simeq \left(1 - \frac{2M}{r}\right)^{1/2} \frac{\partial}{\partial r}, \tag{3.4a}$$

$$\mathbf{e}_{\hat{\theta}} \simeq \frac{1}{r}\frac{\partial}{\partial r}, \tag{3.4b}$$

and

$$\mathbf{e}_{\hat{\varphi}} \simeq \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}. \tag{3.4c}$$

In 3+1-electrodynamics the field tensor $F^{\mu\nu}$ splits into the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ which reside in hyperspace and evolve with the time. If we fix the observer as the FIDO at the given point around a Schwarzschild black hole, the Maxwell equations for the ionosphere are given by (TPM, eq. [2.10])

$$\nabla \cdot \mathbf{E} = 4\pi\rho_e, \tag{3.5a}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{3.5b}$$

$$\nabla \times (\alpha\mathbf{E}) = -\dot{\mathbf{B}}, \tag{3.5c}$$

and

$$\nabla \times (\alpha\mathbf{B}) = \dot{\mathbf{E}} + 4\pi\alpha\mathbf{j}. \tag{3.5d}$$

2. Magnetohydrodynamic Wave Equations in the Ionosphere

The continuity equation and Euler equation of plasma are given by

$$\frac{1}{\alpha}\dot{\rho} + \nabla \cdot (\rho\mathbf{v}) = 0, \tag{3.6a}$$

and

$$\rho\frac{1}{\alpha}\dot{\mathbf{v}} = -\nabla P + \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B}, \tag{3.6b}$$

respectively, where ρ is the density, $\mathbf{v}$ is the velocity, and P is the internal pressure of plasma in the ionosphere. Here we neglect gravitational acceleration which is equal to $-\nabla\alpha/\alpha$.

As usual, we consider small perturbations such as,

$$\rho = \rho_0 + \rho_1, \tag{3.7a}$$

$$P = P_0 + P_1, \tag{3.7b}$$

$$\mathbf{v} = \mathbf{v}_1, \tag{3.7c}$$

and

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1, \tag{3.7d}$$

where the 0-suffixed quantities are the unperturbed and the 1-suffixed are small perturbations. For simplicity we assume that the unperturbed quantities are constant, which can be justified due to the stationarity of the M-T model. In equation (3.7) we do not consider $\mathbf{E}$-perturbation because the electric force can be completely negligible under MHD approximation.

Substituting equation (3.7) into equations (3.5) and (3.6), we get the first-order equations

$$\frac{1}{\alpha}\dot{\rho}_1 + \rho_0 \nabla \cdot \mathbf{v}_1 = 0, \tag{3.8a}$$

$$\rho_0 \frac{1}{\alpha}\dot{\mathbf{v}}_1 + v_s^2 \nabla \rho_1 - \frac{1}{4\pi}(\nabla \times \mathbf{B}_1) \times \mathbf{B}_0 = 0, \tag{3.8b}$$

and

$$\frac{1}{\alpha}\dot{\mathbf{B}}_1 - \nabla \times \mathbf{v}_1 \times \mathbf{B}_0 = 0, \tag{3.8c}$$

where we define speed of sound as

$$v_s^2 \equiv \frac{\gamma P_0}{\rho_0}. \tag{3.8d}$$

3. Plane Wave Solutions in the Ionosphere

A FIDO observes plasma based on the orthonormal basis of his inertial frame, equation (3.4). Therefore, we have

$$\mathbf{B}_0 = \mathbf{B}^T + \mathbf{B}^P \equiv B_0^{\hat{r}} \mathbf{e}_{\hat{r}} + B_0^{\hat{\theta}} \mathbf{e}_{\hat{\theta}} + B_0^{\hat{\varphi}} \mathbf{e}_{\hat{\varphi}} \tag{3.9}$$

Let the angle between $\mathbf{B}_0$ and $\mathbf{e}_{\hat{r}}$ be χ which is set by

$$\chi = \cos^{-1}\frac{1}{B_0}\mathbf{B}_0 \cdot \mathbf{e}_{\hat{r}} = \cos^{-1}\frac{1}{2\pi r B_0 \sin\theta}\mathbf{e}_{\hat{\theta}} \cdot \nabla\Psi. \tag{3.10}$$

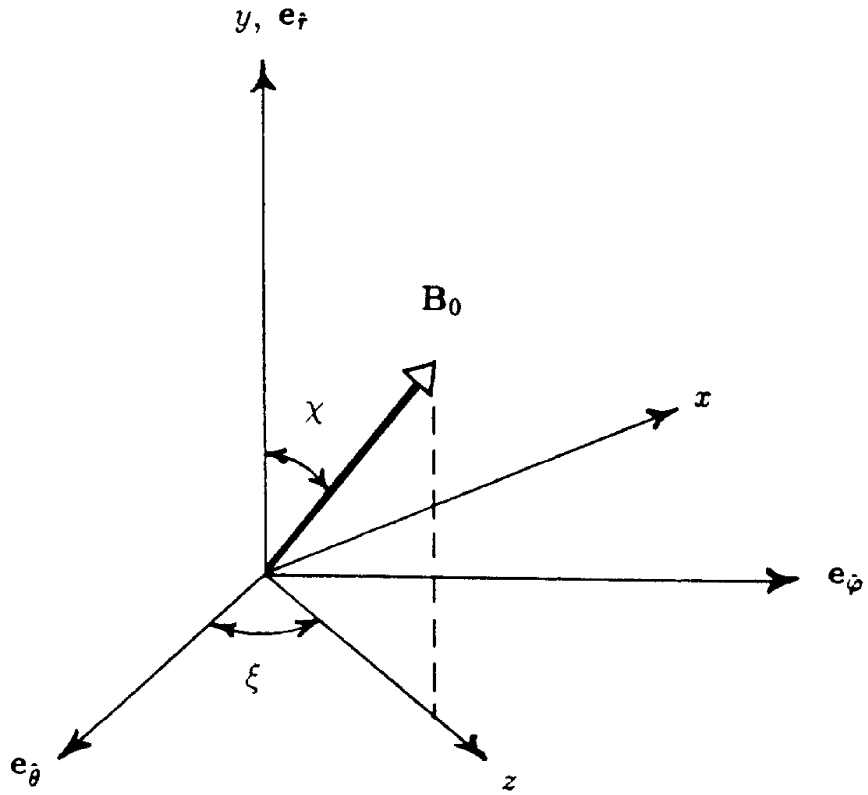

Figure 1. The definitions of the angles ξ and χ

With fixed $\mathbf{e}_{\hat{t}}$, the orthonormal vector in time-axis, we employ another spatial orthonormal basis of the rectangular coordinate (x, y, z) for our FIDO to analyze equation (3.9) more easily. Let the new y-axis coincide with the r-axis and $\mathbf{B_0}$ lie in the y-z plane (see Fig. 1). Let the angle between the θ-axis and the new z-axis be ξ which is set by

$$\xi = \tan^{-1}\frac{B^{\hat{\varphi}}}{B^{\hat{\theta}}} = \tan^{-1}\left[\left(1-\frac{2M}{r}\right)^{-1/2}\frac{4\pi I}{\mathbf{e}_{\hat{r}}\cdot\nabla\Psi}\right], \tag{3.11}$$

Therefore, we have

$$\mathbf{e}_{\hat{x}} = \mathbf{e}_{\hat{\varphi}}\cos\xi + \mathbf{e}_{\hat{\theta}}\sin\xi, \tag{3.12a}$$

$$\mathbf{e}_{\hat{y}} = \mathbf{e}_{\hat{r}}, \tag{3.12b}$$

and

$$\mathbf{e}_{\hat{z}} = -\mathbf{e}_{\hat{\varphi}}\sin\xi + \mathbf{e}_{\hat{\theta}}\cos\xi, \tag{3.12c}$$

We also have

$$\mathbf{v_1} = v_1^{\hat{x}}\mathbf{e}_{\hat{x}} + v_1^{\hat{y}}\mathbf{e}_{\hat{y}} + v_1^{\hat{z}}\mathbf{e}_{\hat{z}} \tag{3.13a}$$

and

$$\mathbf{B_1} = B_1^{\hat{x}}\mathbf{e}_{\hat{x}} + B_1^{\hat{y}}\mathbf{e}_{\hat{y}} + B_1^{\hat{z}}\mathbf{e}_{\hat{z}}. \tag{3.13b}$$

As a trial solution, let us suppose that plane waves can propagate in two directions like the Alfvén waves. Let all the variables depend on z and τ only. Then, we can decompose equation (3.9) into their components. Allowing derivatives with respect to z and τ only, we obtain

$$\frac{\partial}{\partial\tau}B_1^{\hat{x}} = B_0\cos\chi\frac{\partial}{\partial z}v_1^{\hat{x}}, \tag{3.14a}$$

$$\frac{\partial}{\partial\tau}v_1^{\hat{x}} = \frac{B_0\cos\chi}{4\pi\rho_0}\frac{\partial}{\partial z}B_1^{\hat{x}}, \tag{3.14b}$$

$$\frac{\partial}{\partial\tau}B_1^{\hat{y}} = B_0\cos\chi\frac{\partial}{\partial z}v_1^{\hat{y}} - B_0\sin\chi\frac{\partial}{\partial z}v_1^{\hat{z}}, \tag{3.14c}$$

$$\frac{\partial}{\partial\tau}v_1^{\hat{y}} = \frac{B_0\cos\chi}{4\pi\rho_0}\frac{\partial}{\partial z}B_1^{\hat{y}}, \tag{3.14d}$$

$$\frac{\partial}{\partial\tau}v_1^{\hat{z}} = -\frac{v_s^2}{\rho_0}\frac{\partial}{\partial z}\rho_1 - \frac{B_0\sin\chi}{4\pi\rho_0}\frac{\partial}{\partial z}B_1^{\hat{y}}, \tag{3.14e}$$

and

$$\frac{\partial}{\partial\tau}\rho_1 = -\rho_0\frac{\partial}{\partial z}v_1^{\hat{z}}. \tag{3.14f}$$

Equations (3.14a) and (3.14b) yield

$$\frac{\partial^2}{\partial\tau^2}B_1^{\hat{x}} = \frac{B_0^2\cos^2\chi}{4\pi\rho_0}\frac{\partial^2}{\partial z^2}B_1^{\hat{x}}, \tag{3.15a}$$

which shows that $v_1^{\hat{x}}$ and $B_1^{\hat{x}}$ propagate with velocity

$$v_A \cos\chi \equiv \frac{B_0 \cos\chi}{\sqrt{4\pi\rho_0}}. \tag{3.15b}$$

Equations (3.14*c*), (3.14*d*), (3.14*e*), and (3.14*f*) are simultaneous equations of 4 unknowns. Seeking plane wave solutions

$$\sim e^{i(\omega\tau - kz)} \tag{3.16}$$

for all 4 unknowns, we may replace the operators $\partial/\partial\tau$ and $\partial/\partial z$ by $i\omega$ and $-ik$. Then we obtain the system of linear homogeneous equations

$$\begin{pmatrix} -\omega & -kB_0\cos\chi & kB_0\sin\chi & 0 \\ kB_0\cos\chi/4\pi & \rho_0\omega & 0 & 0 \\ -kB_0\cos\chi/4\pi & 0 & \rho_0\omega & -kv_s^2 \\ 0 & 0 & -\rho_0 k & \omega \end{pmatrix} \begin{pmatrix} B_1^{\hat{y}} \\ v_1^{\hat{y}} \\ v_1^{\hat{z}} \\ \rho_1 \end{pmatrix} = 0. \tag{3.17}$$

For this equation to have a solution, the determinant of the 4×4 matrix must be zero. Therefore, we have

$$U^2 - (v_A^2 + v_s^2)U^2 + v_A^2 v_s^2 \cos^2\chi = 0, \tag{3.18a}$$

where U is the phase velocity

$$U \equiv \frac{\omega}{k}. \tag{3.18b}$$

If $\chi = 0$, we have two types of wave, $U^2 = v_A^2$ and $U^2 = v_s^2$. If $\chi = \pi/2$, we have only one type of wave, $U^2 = v_A^2 + v_s^2$. Rewriting the dispersion relation (3.18) as

$$(U^2 - v_A^2)(U^2 - v_s^2) = v_A^2 v_s^2 \sin^2\chi, \tag{3.19}$$

we notice that one root of this equation exceeds the greater of v_A^2 and v_s^2 and the other root is less than the smaller of v_A^2 and v_s^2.

Equations (3.14) are not very different from those in nonrelativistic case except they are redshifted. In general, therefore, there are three distinct types of waves — the Alfvén wave and two magnetosonic waves.

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