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# A NOTE ON MAXIMAL SUBGROUPS IN FINITE GROUPS

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It is well known that if a finite group $G$ has exactly one maximal subgroup then $|G|$ is divisible by one prime only and $G$ is cyclic. In this connection one might ask whether if $G$ has exactly two or three maximal subgroups the above result could be extended. If $G$ has exactly three maximal subgroups then neither $G$ needs to cyclic nor it is required for $|G|$ to be divisible by three primes. Klein 4–group $V$ is an example. However, in the other case, it is shown here that $G$ is indeed divisible by two primes only and $G$ is cyclic. Using this fact, it is proved further that if a group $G$ has exactly two ith maximal subgroups then all the Sylow subgroups of $G$ are cyclic and therefore $G$ is supersolvable. One may recall that $X_i$ is an ith maximal subgroup of a group $G$ if there exists a series $X_0 = G \supset X_1 \supset X_2 \supset \cdots \supset X_i$ of subgroups where $X_k$ is a maximal subgroup of $X_{k-1}$, $1 \leq k \leq i$. All groups considered in this note are finite.

**Lemma 1**. *If a group $G$ has exactly two maximal subgroups then $G$ is nilpotent.*

*Proof.* Let $M$ and $M^*$ be the two maximal subgroups of $G$. If $M \ntrianglelefteq G$ then $N_G(M) = M$ since $M$ is maximal and $[G : N_G(M)] \geq 2$. However $[G : N_G(M)]$ is also the number of conjugates of $M$ in $G$ and since conjugate of a maximal subgroup is again a maximal subgroup it follows that $[G : N_G(M)] = 2$ i.e. $[G : M] = 2$. This implies $M \trianglerighteq G$. Similarly $M^* \trianglerighteq G$. Since all the maximal subgroups of $G$ are normal it follows that $G$ is nilpotent.

**Lemma 2**. *There exists no p-group which has exactly two maximal subgroups.*

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*Proof.* Suppose the assertion is false. Then $S = \{X | X$ is a $p$–group and $X$ has exactly two maximal subgroups $\} \neq \emptyset$. Claim that if $X \in S$ then $X$ is cyclic. Suppose the claim is false, then $S_1 = \{Y | Y \in S$ and $Y$ is not cyclic $\} \neq \emptyset$. Let $Y_0$ be an element of $S_1$ of least possible order. Therefore, if $T \in S \backslash S_1$ then $T$ is cyclic. Suppose $M$ and $M^*$ are the two maximal subgroups of $Y_0$ and $|Y_0| = p^n$. We need to consider two cases : Case I. $M \cap M^* =< e >$, Case II. $M_1 \cap M^* \neq< e >$.

**Case I.** $M \cap M^* =< e >$. Since $M$ and $M^*$ are each normal in $G$, it follows that $Y_0 = M \cdot M^*$ and $|Y_0| = |M| \cdot |M^*|$. Thus $p^n = p^{n-1} \cdot p^{n-1}$ and $p^n = p^2$. Therefore $Y_0$ is elementary abelian since it is not cyclic. If $a$ and $b$ are elements of $Y_0$ then $< a >, < b >, < ab >$ are all maximal subgroups of $Y_0$ and we have a contradiction. Hence in this case it follows that $S_1$ must be empty.

**Case II.** $M \cap M^* \neq< e >$. Let $T = M \cap M^*$ and observe that $T \trianglelefteq G$. Now consider $Y_0/T$. It is a $p$–group, $M/T$, $M^*/T$ are two maximal subgroups of $Y_0/T = \overline{Y}_0$ and $\overline{Y}_0$ does not have any other maximal subgroup besides $M$ and $\overline{M}^*$, since $|\overline{Y}_0| < |Y_0|$ it follows that $\overline{Y}_0 \in S \backslash S_i$. Hence $\overline{Y}_0$ is cyclic, i.e. $\overline{Y}_0 =< \overline{x} >$ for some $\overline{x} \in \overline{Y}_0$. This implies $Y_0 =< x, T >=< x >$ since $T$ is indeed the Frattini subgroup of $Y_0$ and we again have a contradiction. Therefore $S_1$ must be empty and every element in $S$ must be cyclic. If $X \in X$ then $X$ has got exactly one subgroup $L$ of index $p$ which must a maximal subgroup of $X$. Any other subgroup of $X$ will be contained in $L$ and therefore $X$ cannot have another maximal subgroup. Thus $S$ must be empty and this proves the assertion in the lemma.

We omit the proof of the following well known result.

**Lemma 3.** *If a group $G$ has exactly one maximal subgroup then $G$ is a cyclic $p$-group.*

**Theorem 1.** *Let $G$ be a group which has only two maximal subgroups. Then $G$ is cyclic and $|G|$ is divisible by two distinct primes.*

*Proof.* By Lemma 1, $G$ is nilpotent and $G = P_1 \times P_2 \times \cdots \times P_m$, where $P_i$ is the Sylow $p_i$–subgroup of $G$. We claim that $m = 2$. Suppose $m > 2$ and consider $P_i$ (note $m \neq 1$ by Lemma 2). If $P_i$ does not have a proper subgroup then $P_i$ is cyclic of prime order and if $P_i$ has proper subgroup we may conclude that $P_i$ has some maximal subgroup $L_i$. In either of the cases $G$ has a maximal subgroup $H = P_1 \times P_2 \times \cdots \times P_{i-1} \times P_{i+1} \times \cdots \times P_m$ in the first case and $H = P_1 \times P_2 \times \cdots \times P_{i-1} \times L_i \times P_{i+1} \times \cdots \times P_m$ in the second case. Thus for each $i$ there is a maximal subgroup of $G$ and therefore $m = 2$ and $G = P_1 \times P_2$. Evidently, neither $P_1$ nor $P_2$ can have more than one maximal subgroup. Consequently, it implies that $P_1$ and $P_2$ are both cyclic and therefore $G$ is cyclic and the theorem is proved.

The following well known theorem due to B. Huppert is used in the proof of Theorem 3. We mention it here for the sake of completeness.

**Theorem 2**. *If every maximal subgroup of a group $G$ is supersolvable then $G$ is solvable.*

**Lemma 4**. *If every Sylow subgroup of a group $G$ is cyclic then $G$ is supersolvable.*

*Proof.* By induction every maximal subgroup of $G$ is supersolvable and hence $G$ is solvable. Let $N$ be a minimal normal subgroup of $G$. Then $N$ is cyclic and therefore has prime order. Consider $G/N$. It is supersolvable by induction and so $G$ is supersolvable.

**Theorem 3**. *If a group $G$ has only two ith maximal subgroups for some integer $i$ then the Sylow subgroups of $G$ are cyclic and $G$ is supersolvable.*
*Proof.* Let $M_k$ denote a $k$th maximal subgroup of $G$ and consider an $(i-1)$th maximal subgroup $M_{i-1}$. Either $M_{i-1}$ has no proper subgroup in which case $M_{i-1}$ is cyclic of prime order or else $M_{i-1}$ has at most two maximal subgroups. In either of the cases $M_{i-1}$ is cyclic. If $M_{i-2}$ is an $(i-2)$th maximal subgroup then it now follows that each one of its Sylow subgroup must be cyclic and therefore $M_{i-2}$ is supersolvable. Let $M_{i-j}$ be an $(i-j)$the maximal subgroup and assume that all the Sylow subgroups of $M_{i-j}$ are cyclic. Every Sylow subgroup of an $(i-j-1)$th maximal subgroup $M_{i-j-1}$ is contained in some $(i-j)$th maximal subgroup and so is cyclic. It now follows by induction that every Sylow subgroup of $G$ is cyclic and by lemma 3 is supersolvable and the proof is complete.

It was remarked earlier that if a group $G$ possesses exactly three maximal subgroups then the order of $G$ need not be divisible by three primes, $G$ could be a $p$-group. In fact for such a $p$-group the prime $p$ must be 2.

**Proposition**. *There is no p-group for odd p with exactly three maximal subgroups.*

*Proof.* Let $P_1, P_2, P_3$ be three maximal subgroups of a $p$-group $P, p \neq 2$ and $|P| = p^n$. If $P_1 \cap P_2 =< e >$ then $|P_1P_2| = |P|$ implies $p^n = p^2$ so that $\frac{p^2-1}{p-1} = 3$ and $p = 2$, a contradiction. Suppose $N = P_1 \cap P_2$ and consider $P/N$. If $N \not\subseteq P_3$ and $P/N$ is a $p$-group with exactly two maximal subgroup $P_1/N$ and $P_2/N$ which is impossible. Thus $N = P_1 \cap P_2 \cap P_3$ and

$|P/N| = p^{n^*}$ for some integer $n^*$. Since two maximal subgroups $P_1/N$ and $P_2/N$ of $P/N$ intersect trivially, $p^{n^*} = p^2$ and once again we get $p = 2$, a contradiction. Therefore the assumption that the proposition is false is wrong and the proof is complete.

The nonabelian 2–groups are all classified [Thm.14.9 p.91, [1]]. Evidently, abelian groups having exactly three maximal subgroups can have the order divisible by at most three primes. In fact a group with three maximal subgroups which is not a $p$–group must be cyclic and its order is divisible by three primes.

**Theorem 4.** *A group $G$ which has exactly three maximal subgroups and is not a group of prime power order is necessarily cyclic and its order is divisible by at most three primes.*

*Proof.* Let $M_1, M_2, M_3$ be the maximal subgroups of $G$. If none of $M_i \trianglelefteq G$, $i = 1, 2, 3$ then $N_G(M_1) = M_1$ and $[G : N_G(M_1)] =$ number of conjugates of $M_1 = 3$. For if $[G : N_G(M_1)] = 2$ then $[G : M_1] = 2$ and so $M_1 \trianglelefteq G$. This however implies the indices of all the maximal subgroups are same. If $p | [G : N_G(M_1)] = [G : M_1]$ then there is a maximal subgroup containing a Sylow $p$–subgroup and its index is prime to $p$. Consequently, the index $M_1$ is not divisible by $p$ and we have a contradiction. Hence $M_1 \trianglelefteq G$. This implies $M_2 \trianglelefteq G$ as otherwise, $[G : N_G(M_2)] = [G : M_2] =$ number of conjugate of $M_2 = 2$ unless $M_3 \trianglelefteq G$. If $M_3 \trianglelefteq G$ then $M_2$ is not conjugate of $M_3$ or $M_1$ and therefore $M_2 \trianglelefteq G$. If on the other hand $[G : M_2] = 2$, $M_2 \trianglelefteq G$ and again $M_3$ then is necessarily normal in $G$. Thus all the Sylow subgroups of $G$ are normal and $G = P_1 \times P_2 \times \cdots \times P_m$ where $P_i$ is a Sylow $p_i$–subgroup of $G$. Note $m \leq 3$, as otherwise $G$ will have more than three maximal subgroups. Thus $|G|$ is divisible by at most three primes and $m = 2$ or 3. However if $m = 2$ i.e. $G = P_1 \times P_2$ then $G$ will have either less than three or more than three maximal subgroups. This follows easily from the consideration of maximal subgroups of $P_1$ and $P_2$. Thus $m = 3$ and $P_i$ has at most one maximal subgroup $i = 1, 2, 3$. This however implies that $G$ is cyclic and the proof is complete.

**Corollary.** *A group $G$ which has exactly three second maximal subgroups is solvable.*

*Proof.* If $M$ is any maximal subgroup of $G$ then $M$ has at most three maximal subgroups and therefore $M$ is either cyclic or is a 2-group. Thus every maximal subgroup of $G$ is supersolvable and consequently $G$ is solvable.

*Remark.* A group satisfying the condition in the corollary above will have cyclic Sylow subgroups corresponding to odd prime divisors of the group order. $A_4$ is an example of such group.

# References

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