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SPAN OF PRODUCT DOLD MANIFOLDS

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By using γ-operation in KO-theory, we estimate the upperbound of the number of linearly independent tangent vector fields on a product Dold manifold. As corollaries, we have similar results for a product of two real or complex projective spaces.

1. Introduction

The Dold manifold $D(m,n)$ of dimension $m+2n$ is defined as the quotient manifold of $S^m \times CP(n)$ by identifying (x,z) with $(-x,\overline{z})$, where $\overline{z}$ is the complex conjugate. $D(m,0)$ and $D(0,n)$ are readily seen to be the real projective space $RP(m)$ and the complex projective space $CP(n)$, respectively.

In [8], J.J. Ucci determined the stable tangent bundle in terms of two canonical line bundles and the Grothendieck rings $K(D(m,n))$ and $KO(D(m,n))$ for the Dold manifold $D(m,n)$. He applied them to the problem of non–immersion and nonembedding for the manifold $D(m,n)$ using the methods initiated by M.F. Atiyah [1].

In this paper we shall derive an upper bounded of span of product Dold manifold.

Let $\phi(m)$ be the number of integer s with $0 < s \leq m$ and $s \equiv 0,1,2$ or 4 mod 8

$$\delta^*(m,n) = \begin{cases} \max\{s, 2[\frac{n}{2}] \,|\, 2^{s-1}\binom{m+n+1}{s} \neq 0 \bmod 2^{\phi(m)} \text{ if } m \neq 0 \\ 2[\frac{n}{2}] \qquad \text{if } m = 0. \end{cases}$$

where $[\frac{n}{2}]$ denotes the integer part of $\frac{n}{2}$.
Note that $\delta^*(m,0) = \delta(m)$ is defined in [7].

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Then our result can be stated as follows :

Theorem 3.2. *The number of linearly independent fields of tangent vectors on* $D(m,n) \times D(u,v)$ *does not exceed* $m+2n+u+2v - \max\{\delta^*(m,n), \delta^*(u,v)\}$.

Theorem 3.2 can be viewed as an extension of Corollary 4.3 in H. Suzuki [7]. In what follows, M will mean a smooth closed manifold. By immersion and embedding we will mean C^∞–differentiable ones. A positive integer $\imath$ will denote either itself or the trivial $\imath$–plane bundle over an appropriate space and $\imath\xi = \xi \oplus \xi \oplus \cdots \oplus \xi$ ($\imath$–times Whitney sum).

2. Known results for the Dold manifolds

Let $\widetilde{\xi}$ be the canonical real line bundle over $RP(m)$, and let $\widetilde{\eta}$ be the canonical complex line bundle over $CP(n)$. Define a line bundle ξ over $D(m,n)$ as follows : the total space $E(\xi)$ of ξ is obtained from $S^m \times CP(n) \times R$ by identifying (x,z,t) with $(-x,\overline{z},-t)$. For $n=0$, ξ is just the canonical line bundle $\widetilde{\xi}$ over $D(m,0)=RP(m)$ which implies $i^*\xi = \widetilde{\xi}$. We define another real 2–plane bundle ξ over $D(m,n)$ whose total space $E(\eta)$ is $S^m \times S^{2n+1} \times C$ mudulo the identification $(x,\rho,w) \sim (x,\lambda\rho,\lambda w) \sim (-x,\overline{\lambda\rho},\overline{\lambda,w}) \sim (-x,\overline{\rho},\overline{w})$, $\lambda \in S^1 \subset C$. For $m=0$, η is just the canonical complex line bundle $\widetilde{\eta}$ over $D(0,n)=CP(n)$ considered as a real bundle (denoted by $\mathrm{re}(\widetilde{\eta})$) which implies $j^*\eta = \mathrm{re}\ (\widetilde{\eta})$.

Now we can describe the tangent bundle $\tau(D(m,n))$ of the Dold manifold $D(m,n)$ [8] :

Theorem 2.1. $\tau(D(m,n)) \oplus \xi \oplus 2 = (m+1)\xi \oplus (n+1)\eta$.

Let F denote either the real field R or the complex field C and let $\mathrm{Vect}_F(M)$ denote the set of isomorphism classes of F-vector bundles over M. The Whitney sum of vector bundles makes $\mathrm{Vect}_F(M)$ a semi-group and the Grothendieck group $K_F(M)$ is the associated abelian group. The tensor product of vector bundles defines a commutative ring structure in $K_F(M)$. As usual, we use the notation $KO(M)$ and $K(M)$ for $K_R(M)$ and $K_C(M)$, respectively. Let x_0 be a base point of M, then clearly $KO(x_0)=Z$. We define $\widetilde{KO}(M) = \ker\ \{i^* : KO(M) \to Z\}$, where i^* is the homomorphism induced by the natural inclusion $\{x_0\} \to M$, then clearly $KO(M) \approx Z \oplus \widetilde{KO}(M)$. We write $x = \xi - 1$, $y = \eta - 2 - x$. In [8], J. Ucci has also computed the Grothendieck ring $\widetilde{KO}(D(m,n))$.

Theorem 2.2. $\widetilde{KO}(D(m,n))$ *contains a summand isomorphic to* $Z_{2^{\phi(m)}} \oplus$

$Z^{[\frac{n}{2}]}$ generated by $x, y, y^2, \cdots, y^{[\frac{n}{2}]}$ with the relation $2^{\phi(m)}x = 0$. The multiplicative structure $Z_{2^{\phi(m)}} \oplus Z^{[\frac{n}{2}]}$ is given by $x^2 = -2x$ and $xy = 0$. Moreover, it can be shown that $y^{[\frac{n}{2}]+1}$ vanishes.

For $x \in \mathrm{Vec}_R(M)$, the vector bundle $\lambda^i(x)$ is defined by the exterior power operation $\Lambda^i(x), i = 0, 1, 2, 3, \cdots$. We define $\lambda_t(x) = \sum_{i=0}^{\infty} \lambda^i(x)t^i$, where t is an indeterminate. Let $A(M)$ denote the multiplicative group of formal power series in t with coefficient in $KO(M)$ and with constant term 1. Then λ_t is a homomorphism $\mathrm{Vect}_R(M) \to A(M)$. Hence we get a homomorphism $\lambda_t : KO(M) \to A(M)$ and operators $\lambda^i : KO(M) \to KO(M)$ with $\lambda_t(x) = \sum_{i=0}^{\infty} \lambda^i(x)t^i$. The γ-operation in $KO(M)$, $\gamma_t : KO(M) \to A(M)$, is defined by the requirement that $\gamma_t(x) = \lambda_{\frac{t}{1-t}}(x)$ and $\gamma_t(x) = \sum_{i=0}^{\infty} \gamma^i(x)t^i$ for $x \in KO(M)$. Now let $\tau(M)$ denote the tangent bundle over M and put $\tilde{\tau}(M) = \tau(M) - \dim(M) \in \widetilde{KO}(M)$, the operation r^i gives us an information about the structure of tangent bundle on M as follows:

Theorem 2.3 *If we have $r^i(\tilde{\tau}) \neq 0$ for an i such that $0 \leq i \leq n$, then the number of linearly independent fields of tangent vectors on M does not exceed $n - i$.*

Let $D(m,n), D(u,v)$ be the Dold manifolds and let

$$\begin{aligned} \Pi_1 &: D(m,n) \times D(u,v) \to D(m,n), \\ \Pi_2 &: D(m,n) \times D(u,v) \to D(u,v), \\ \Pi_\wedge &: D(m,n) \times D(u,v) \to D(m,n) \wedge D(u,v), \end{aligned}$$

be the canonical projection, where $D(m,n) \wedge D(u,v)$ is the Smash product of the Dold manifolds $D(m,n)$ and $D(u,v)$. The following comes from (2.3) of H. Suzuki [7].

Theorem 2.4. (i) *The induced homomorphisms*

$$\begin{aligned} \Pi_1^* &: \widetilde{KO}(D(m,n)) \to \widetilde{KO}(D(m,n) \times D(u,v)), \\ \Pi_2^* &: \widetilde{KO}(D(u,v)) \to \widetilde{KO}(D(m,n) \times D(u,v)), \\ \Pi_\wedge^* &: \widetilde{KO}(D(m,n) \wedge D(u,v)) \to \widetilde{KO}(D(m,n) \times D(u,v)), \end{aligned}$$

are injective and we have a direct sum decomposition

$$\begin{aligned} \widetilde{KO}(D(m,n) \times D(u,v)) &= \Pi_1^*(\widetilde{KO}(D(m,n)) \oplus \Pi_2^*(\widetilde{KO}(D(u,v)) \\ &\quad \oplus \Pi_\wedge^*(\widetilde{KO}(D(m,n) \wedge D(u,v))). \end{aligned}$$

(ii) *If* $\delta \in \widetilde{KO}(D(m,n))$ *and* $\xi \in \widetilde{KO}(D(u,v))$, *then* $\Pi_i^*(\delta)\Pi_2^*(\zeta) \in \Pi_\wedge^*(\widetilde{KO}(D(m,n)\wedge D(u,v)))$.

3. Linearly independent tangent vector fields on a product Dold manifolds

Put $\tau_1 = \tau(D(m,n))$. Using Theorem 2.1, we have

$$\begin{aligned}\widetilde{\tau_1} &= \tau_1(D(m,n)) - (m+2n)\\ &= (m+1)\xi + (n+1)\eta - \xi - 2 - m - 2n\\ &= (m+n+1)x + (n+1)y\end{aligned}$$

Since γ_t is homomorphism, $\gamma_t(\widetilde{\tau_1}) = \gamma_t(x)^{(m+n+1)}\ \gamma_t(y)^{(n+1)}$ and since $\gamma_t(x) = 1 + xt$, $\gamma_t(y) = 1 + yt - t^2$, $y^{[\frac{n}{2}]+1} = 0$, $x^2 = -2x, xy = 0$.

$$\begin{aligned}\gamma_t(\widetilde{\tau_1}) &= (1+xt)^{m+n+1}(1+yt-yt^2)^{n+1}\\ &= \{\sum_{i=0}^{m+n+1}\binom{m+n+1}{i}x^it^i\}\{\sum_{j=0}^{n+1}\binom{n+1}{j}y^j(t-t^2)^j\}\\ &= \{1+\sum_{i=1}^{m+n+1}\binom{m+n+1}{i}x^it^i\}\{1+\sum_{i=1}^{n+1}(\sum_{j=[\frac{i+1}{2}]}^{i}\alpha_{ij}y^j)t^i\}\\ &= 1+\sum_{i=1}^{m+n+1}\{(-2)^{i-1}\binom{m+n+1}{i}x+\sum_{j=[\frac{i+1}{2}]}^{i}\alpha_{ij}y^i\}t^i\end{aligned}$$

where α_{ij} is integers and $\alpha_{ij} = 0$ for $i > n+1$, and the coefficient of t^s is

$$\gamma^s(\widetilde{\tau_1}) = (-2)^{i-1}\binom{m+n+1}{i}x + \sum_{j=[\frac{i+1}{2}]}^{i}\alpha_{ij}y^j.$$

Now let $\tau_2 = \tau(D(u,v))$ be the tangent bundles and let μ, λ be the line and real 2–plane bundle over $D(u,v)$ respectively, and let $z = \mu - 1$, $w = \lambda - 2 - z$. Then

$$\begin{aligned}\widetilde{\tau} &= \tau(D(m,n)\times D(u,v)) - (m+2n+u+2v)\\ &= \pi_1^*(\tau_1) + \pi_2^*(\tau_2) - (m+2n+u+2v)\\ &= \pi_1^*((m+n+1)x+(n+1)y) + \pi_2^*((u+v+1)z+(v+1)w)\end{aligned}$$

By using the property $\gamma_t(x+y)=\gamma_t(x)\gamma_t(y)$ and the naturality of operator γ_t, we have

$$\begin{aligned}\gamma_t(\tilde{\tau}) &= \pi_1^*(\gamma_t(x)^{(m+n+1)}\gamma_t(y)^{n+1})\pi_2^*(\gamma_t(z)^{(u+v+1)}\gamma_t(w)^{v+1})\\ &= (1+\pi_1^*(x)t)^{m+n+1}(1+\pi_1^*(y)(t-t^2))^{n+1}(1+\pi_2^*(z)t)^{u+v+1}\\ &\quad (1+\pi_2^*(w)(t-t^2))^{v+1}\\ &= [1+\sum_{i=1}^{m+n+1}\{(-2)^{i-1}\binom{m+n+1}{i}\pi_1^*(x)+\sum_{j=[\frac{i+1}{2}]}^{i}\alpha_{ij}\pi_1^*(y)^j\}t^i]\\ &\quad [1+\sum_{k=1}^{u+v+1}\{(-2)^{k-1}\binom{u+v+1}{k}\pi_2^*(z)+\sum_{\ell=[\frac{k+1}{2}]}^{k}\beta_{k\ell}\pi_2^*(w)^\ell\}t^k]\end{aligned}$$

and so the coefficient of t^s is

$$\begin{aligned}\gamma^s(\tilde{\tau}) &= (-2)^{s-1}\binom{m+n+1}{s}\pi_1^*(x)+\sum_{j=[\frac{s+1}{2}]}^{s}\alpha_{sj}\pi_1^*(y)^j\\ &\quad +\sum_{a+c=s,1\le a\le s-1}\{(-2)^{a-1}\binom{m+n+1}{a}\pi_1^*(x)\sum_{b=[\frac{a+1}{2}]}^{a}\alpha_{ab}\pi_1^*(y)^b\}\\ &\quad \{(-2)^{c-1}\binom{u+v+1}{c}\pi_2^*(z)+\sum_{d=[\frac{c+1}{2}]}^{c}\beta_{cd}\pi_2^*(w)^d\\ &\quad +(-2)^{s-1}\binom{u+v+1}{c}\pi_2^*(z)+\sum_{\ell=[\frac{s+1}{2}]}^{s}\beta_{s\ell}\pi_2^*(w)^\ell.\end{aligned}$$

Therefore, we have

Lemma 3.1. *If $\gamma^s(\tilde{\tau}_1)\neq 0$ or $\gamma^s(\tilde{\tau}_2)\neq 0$ then $\gamma^s(\tilde{\tau})\neq 0$.*

By using the function $\delta^*(m,n)$ defined in the introduction we have the following results.

Theorem 3.2. *The number of linearly independence fields of tangent vectors on $D(m,n)\times D(u,v)$ does not exceed $m+2n+u+2v-\max\{\delta^*(m,n), d^*(u,v)\}$*

Proof. We put $S_0=\max\{\delta^*(m,n),\delta^*(u,v)\}$. We can easily check that $0\le\delta^*(m,n)\le m+2n$ and $0\le\delta^*(u,v)\le u+2v$. Hence it follows that $0\le S_0\le m+2n+u+2v$.
The cofficient of $t^{2[\frac{n}{2}]}$ is nonzero since $\alpha_{2[\frac{n}{2}],[\frac{n}{2}]}\neq 0$. Using the definition of δ^* and Lemma 3.1, we obtain the result.

Corollary 3.3. *The number of linearly independent fields of tangent vectors on* $\mathbf{R}P(m) \times \mathbf{R}P(u)$ *does not exceed* $m + u - \max\{\delta(m), \delta(u)\}$.

Corollary 3.4. *The number of linearly independent fields of tangent vectors on* $\mathbf{R}P(m) \times \mathbf{C}P(v)$ *does not exceed* $m + 2v - \max\{\delta(m), 2[\frac{v}{2}]\}$.

Corollary 3.5. *The number of linearly independent fields of tangent vectors on* $\mathbf{C}P(n) \times \mathbf{C}P(v)$ *does not exceed* $2n + 2v - \max\{2[\frac{n}{2}], 2[\frac{v}{2}]\}$.

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