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THE STRUCTURE OF TOPOLOGICAL REGULAR SEMIGROUPS

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Throughout, a semigroup will mean a topological semigroup, i.e., a Hausdorff space with a continuous associative multiplication. A semigroup S is termed *regular* if and only if $x \in xSx$ for each x in S [1]. If S is a regular semigroup, then for each $a \in S$, there exists an element $b \in S$ such that $a = aba$ and $b = bab$ (b is called an inverse of a). If b is an inverse of a, then ab and ba are both idempotents but are not always equal. It is quite clear that a regular semigroup S is a group if S has only one idempotent. Moreover, every regular I–semigroup is a semilattice and every almost pointwise periodic I–semigroup is also a semilattice [4]. For an element a of a regular semigroup S, we will adopt the notation

$$V(a) = \{x \in S \mid x \quad \text{is an inverse of} \quad a\}.$$

Let $\leq$ be a quasi–order on a set X. For a subset A of X, the following notations will be standard [2], [3] :

$$L(A) = \{y \in X \mid y \leq x \quad \text{for some} \quad x \in A\},$$

$$M(A) = \{y \in X \mid x \leq y \quad \text{for some} \quad x \in A\},$$

$$I(A) = L(A) \cap M(A).$$

A quasi–order $\leq$ on a topological space X is said to be *continuous* if and only if, whenever $a \not\leq b$ in X, there are open sets U and V, $a \in U$, $b \in V$, such that if $x \in U$ and $y \in V$, then $x \not\leq y$.

By Ward, it has been shown that a quasi–ordered topological space is continuous if and only if the graph of the quasi–order is closed [3]. A subset

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A of a quasi-ordered topological space X is said to be *convex* provided $A = I(A)$. X is said to be *quasi-locally convex* provided, whenever $x \in X$ and $I(x) \subset U$, an open set, there is a convex open set V such that $I(x) \subset V \subset U$.

As a generalization of the theorem obtained by Nachbin [2], Ward showed that a compact Hausdorff quasi-ordered topological space with continuous quasi-order is quasi-locally convex [3].

Lemma 1. *Let $\leq$ be a relation on a regular semigroup S by $x \leq y$ if and only if $x \in Sy$. Then $\leq$ is a continuous quasi-order on S.*

Proof. By using Wallace's theorem, the compactness of S gives the proof immediately.

All regular semigroups concerned are assumed to have this quasi-order. By Lemma 1 together with the fact that $L(x) = Sx$, one obtain the following useful results:

Lemma 2. *Let S be a compact regular semigroup. Then S is quasi-locally convex.*

Lemma 3. *Let S be a regular semigroup and let $a \in V(x)$. Then*

(1) $L(x) = Sx = Sax = L(ax)$, $M(x) = M(ax)$, *and* $I(x) = I(ax)$.

(2) $L(x)ax = L(x), M(x)ax \subset M(x)$, *and* $I(x)ax \subset I(x)$.

Lemma 4. *Let S be a commutative regular semigroup. Then*

(1) $xy \leq x, y$ *for* $x, y \in S$.

(2) *If* $a \leq b$ *and* $c \leq d$, *then* $ac \leq bd$.

(3) ax *is the identity for the ideal* $L(x)$ *of* S.

(4) $I(x) = H(ax)$, *the maximal subgroup of S with identity ax for every $a \in V(x)$.*

(5) *If a and b are inverses of x, then $ax = bx$.*

Proof. (1) and (3) are obvious. To prove (4), let $xa = ax = e$. Then $e^2 = e$. Let $y \in H(e)$. Then $y'y = e$ for some $y' \in H(e)$. Hence

$$y = ye = yax \in Sx = L(x),$$

$$x = xax = xe = xy'y \in Sy, \text{i.e.}, y \in M(x).$$

Now let $y \in I(x)$. Then $y \in eSe$ and $e \in yS \cap Sy$, i.e., $y \in H(x)$.
(5) is immediate from (3) and (4).

Theorem 5. *Let S be a regular semigroup in which idempotents commute. Then*

(1) *$V(a)V(b) \subset V(ba)$, for $a, b \in S$.*
(2) *If $V(a) \cap V(a^n) \neq \emptyset$ for $n > 1$, then $a^n = a$.*
(3) *If $x^n = x$ for $n > 1$ and if $x \in V(a)$, then $a^n = a$.*

Proof. (1) Let $x \in V(a)$, $y \in V(b)$. Since ax and yb are idempotents,

$$(xy)(ba)(xy) = x(yb)(ax)y = x(ax)(yb)y = xy,$$

$$(ba)(xy)(ba) = b(ax)(yb)a = b(yb)(ax)a = ba.$$

(2) Let $x \in V(a) \cap V(a^n)$. Then $axa = a$ and $xa^n x = x$. Since ax and $a^n x$ are idempotents,

$$\begin{aligned} a &= axa = a(xa^n x)a = (ax)(a^n x)a \\ &= (a^n x)(ax)a = a^n xa = a^{n-1}(axa) = a^n. \end{aligned}$$

(3) Since $x \in V(a)$ and since $V(a)^n \subset V(a^n)$ by (1),

$$x = x^n \in V(a)^n \subset V(a^n), \quad \text{i.e.,}$$

$$x \in V(a) \cap V(a^n) \neq \emptyset.$$

Hence $a^n = a$ by (2).

Lemma 6. *Let S be a regular semigroup. Then*
(1) *$V(x)$ is closed for each $x \in S$.*
(2) *$L(x), M(x)$ and $I(x)$ are closed for each $x \in S$ if S is compact.*

Proof. (1) If $y \notin V(x)$, $x \neq xyx$. By the continuity of multiplication, there is an open set W such that $y \in W$ and $x \notin xWx$. Hence $\{y | x = xyx\}$ is closed. Similarly, $\{y | y = yxy\}$ is closed since $y \neq yxy$. Hence

$$V(x) = \{y | x = xyx\} \cap \{y | y = yxy\}$$

is closed. (2) is obvious since S is compact.

Definition. A semigroup S is said to have *small property* at $x \in S$ if and only if, for any open set U about $I(x)$, there is an open subsemigroup V about $I(x)$ contained in U. We say S has small property if it has *small property* at every point of S.

Theorem 7. *Let S be a compact commutative regular semigroup and let $x \in S$. If $L(x)$ has small property at x, then S has small property at x.*

Proof. Let U be an open set containing $I(x)$. Since S is compact, by Lemma 6, $I(x)$ is compact. Then there is an open set P such that $I(x) \subset$

$P \subset P^* \subset U$, where P^* is the closure of P. Since P^* is compact, by Lemma 2, P^* is quasi-locally convex. Then there is a convex open set V in P^* such that $I(x) \subset V \subset W$. Since $L(x)$ has small property at x, there is an open set K in $L(x)$ such that

$$I(x) \subset K \subset V \cap L(x), \quad K^2 \subset K.$$

Let $a \in V(x)$. Define a function $f : S \to L(x)$ by $f(s) = sax$. Then f is a continuous homomorphism.

By Lemma 3, $f^{-1}(K)$ is an open subsemigroup of S containing $I(x)$. Now let $A = f^{-1}(K) \cap V$. Then A is open and $I(x) \subset A \subset U$. To show $A^2 \subset A$, let $b, c \in A$. Then $bcax \in V$. Since V is convex, by Lemma 4, $bc \in V$. Therefore S has small property at x.

If S is a semilattice, then S is a commutative regular semigroup and $\leq$ is a partial order, and hence $I(x) = \{x\}$ for each $x \in X$. Therefore Theorem 7 gives the following corollary which is a result obtained by Lawson :

Corollary 8. *Let S be a locally compact topological semilattice and let $x \in S$. If $L(x)$ has small semilattices at x, then S has small semilattices at x* [5].

Theorem 9. *Let S be a compact commutative regular semigroup in which $I(x)$ is a component of S for each $x \in S$. Then S has small property.*

Proof. If $x \in S$, then $L(x)$ is a compact commutative regular semigroup with the identity ax, where $a \in V(x)$. By Wallace [6],[8], $L(x)$ has small property at x, and Theorem 7 completes the proof.

Theorem 9 is a generalization of the following corollary obtained by Lawson :

Corollary 13. *If S is a locally compact, totally disconnected topological semi-lattice, then S has small semilattices.*

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