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ON PERTURBATIONS OF NONLINEAR INTEGRODIFFERENTIAL EQUATIONS

Thabet Ahmed Abdou and Abdel–Hadi Alem Ahmed

1. Asymptotic Properties of Solutions

In 1961, V.M. Elekseev [1] developed a variation of constants formula which has led to the development of many interesting results on the behaviour of solutions of perturbed nonlinear differential equations. In [3], F. Brauer has developed a formula analogous to the Alekseev formula for solutions of perturbed nonlinear integrodifferential and integral equations. Recently, S.R. Bernfeld and M.E. Lord [2] obtained another variation of constants formula for perturbed nonlinear integrodifferential and integral equations by assuming the existence of the inverse of the fundamental matrix corresponding to the unperturbed systems and applied it to some important problems in the theory of integral equations. In this section we wish to study the behavioural relationships between the solutions of the integrodifferential systems

$$\dot{x}(t) = f(t, x(t), \int_{t_0}^{t} k(t, s, x(s))ds), x(t_0) = x_0, \tag{1.1}$$

and its perturbed system

$$\begin{aligned} \dot{y}(t) &= f(t, y(t), \int_{t_0}^{t} k(t, s, y(s))ds) + g(t, y(t), \int_{t_0}^{t} h(t, s, y(s))ds), \\ y(t_0) &= x_0, \end{aligned} \tag{1.2}$$

by using the variant of the variation of constants formula developed by Bernfeld and Lord in [2] and the integral inequality recently established by B.G. Pachpatte in [7]. In equations (1.1) and (1.2), x, y, k, h, f, g and x_0 are the elements of R^n, an n–dimensional Euclidean space. Let I be the

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interval $0 \leq t < \infty$ and $C[x, y]$ denote the space of continuous functions from X to Y where X and Y are convenient spaces. We shall assume that $k, h \in C[I \times I \times R^n, R^n]$ and $f, g \in C[I \times R^n \times R^n, R^n]$. We use $x(t) = x(t, t_0, x_0)$ to denote the solution of (1.1) passing through the point (t_0, x_0) and $y(t) = y(t, t_0, x_0)$ to denote a solution of (1.2) passing through the point (t_0, x_0) for $t_0 \geq 0$. The symbol $|\cdot|$ will denote some convenient norm on R^n as well as a corresponding consistent matrix norm.

Recently, many results concerning the behaviour of solutions of some special forms of equation (1.1) and (1.2) have been investigated by Brauer [3], Bernfeld and Lord [2], Corduneanu [4,5], Grossman and Miller [6], and Pachpatte[8,9,10,11].

In particular, if we impose on f and g various meanings, it is apparent that equations (1.1) and (1.2) have a great diversity. For example, if f and g in (1.1) and (1.2) are of the form

$$f(t, u, \int_{t_0}^{t} k(t, s, u) ds) = F(t, u) + \int_{t_0}^{t} k(t, s, u) ds,$$

and

$$g(t, u, \int_{t_0}^{t} h(t, s, u) ds) = G(t, u) + H(t, u) + \int_{t_0}^{t} h(t, s, u) ds,$$

then equations (1.1) and (1.2) reduces to the integrodifferential equations considered by Brauer in [3] and Bernfeld and Lord in [2].
In the special case when $k(t, s, u) = 0$ and $h(t, s, u) = 0$, then equations (1.1) and (1.2) reduces to the ordinary differential equations studied by many authors in the literature.

Preliminaries. In our subsequent discussion our interest lies in the following definitions in terms of the behaviour of solutions of (1.1). For similar definitions the reader is referred to [8, 9, 10, 11].

Definition 1.1. The solution $x(t, t_0, x_0)$ of (1.1) is said to be globally uniformly stable if there exists a constant $M > 0$ such that

$$|x(t, t_0, x_0)| \leq M|x_0|, \text{ for all } t \geq t_0 \geq 0 \text{ and } |x_0| < \infty.$$

Definition 1.2. The solution $x(t, t_0, x_0)$ of (1.1) is said to be exponentially asymptotically stable if there exist constants $M > 0, \alpha > 0$ such that $|x(t, t_0, x_0)| \leq M|x_0|e^{-\alpha(t-t_0)}$, for all $t \geq t_0 \geq 0$, $|x_0|$ is sufficiently small.

Definition 1.3. The solution $x(t, t_0, x_0)$ of (1.1) is said to be uniformly slowly growing, if and only if for every $\alpha > 0$ there exists a constant $M > 0$, possibly depending on α, such that $|x(t, t_0, x_0)| \leq M|x_0|e^{-\alpha(t-t_0)}$, for all $t \geq t_0 \geq 0$ and $|x_0| < \infty$.

In order to establish our main results in this section we require the following integral inequality recently established by Pachpatte ;

Lemma 1.1. *(Pachpatte [7]):*

Let $u(t), a(t)$ and $b(t)$ be real-valued nonnegative continuous functions defined on I, for which the inequality

$$u(t) \leq u_0 + \int_{t_0}^{t} a(s)u(s)ds + \int_{t_0}^{t} a(s)(\int_{t_0}^{s} b(\tau)u(\tau)d\tau)ds, t \in I,$$

holds, where u_0 is a nonnegative constant. Then

$$u(t) \leq u_0[1 + \int_{t_0}^{t} a(s) \exp(\int_{t_0}^{s} [a(\tau) + b(\tau)]d\tau)ds], t \in I,$$

Variation of constants formula

In this section we present a slight variant of the nonlinear variation of constants formula developed by Bernfeld and Lord [2] for perturbed integrodifferential system (1.2) which is useful to establish our main results in the next section.

Theorem 1.1. *Suppose that the system*

$$x(t) = f(t) + \int_{t_0}^{t} H(t, s, x(s), \int_{t_0}^{s} k(s, \tau, x(\tau))d\tau)ds \tag{1.3}$$

$x(t_0) = f(t_0)$, admits unique solutions and that $\Phi(t, t_0, x_0)$ exists and is continuous for all $t \geq t_0$ and that $\Phi^{-1}(t, t_0, x_0)$ exists for all $t \geq t_0$. Then any solution $y(t)$ of :

$$\begin{aligned} y(t) &= f(t) + \int_{t_0}^{t} H(t, s, y(s), \int_{t_0}^{s} k(s, \tau, y(\tau)d\tau)ds \\ &\quad + \int_{t_0}^{t} G(t, s, y(s), \int_{t_0}^{s} h(s, \tau, y(\tau))d\tau)ds, y(t_0) = f(t_0) \end{aligned} \tag{1.4}$$

satisfies the relation

$$\begin{aligned} y(t) &= x(t, t_0, x_0 + \int_{t_0}^{t} \Phi^{-1}(s, t_0, v(s))[G(s, s, y(s), \int_{t_0}^{s} h(s, \tau, y(\tau))d\tau) \\ &\quad + \int_{t_0}^{s} G_s(s, \tau, y(\tau), \int_{t_0}^{\tau} h(\tau, u, y(u))du)d\tau]ds), \end{aligned} \tag{1.5}$$

as far as $v(t)$ exists to the right of t_0, and $v(t)$ is determined by

$$\begin{aligned} \dot{v}(t) &= \Phi^{-1}(t,t_0,v(s))[G(t,t,y(t),\int_{t_0}^{t} h(t,\tau,y(\tau))d\tau) \\ &\quad + \int_{t_0}^{t} G_t(t,s,y(s),\int_{t_0}^{s} h(s,\tau,y(\tau))d\tau)ds], v(t_0) = f(t_0). \end{aligned} \tag{1.6}$$

From :

$$\begin{aligned} y(t,t_0,x_0) &= x(t,t_0,x_0) + \int_{t_0}^{t} \Phi(t,t_0,v(s))\Phi^{-1}(s,t_0,v(s)). \\ &\quad \cdot g(s,y(s,t_0,x_0),\int_{t_0}^{s} h(s,\tau,y(\tau,t_0,x_0))d\tau)ds \end{aligned} \tag{1.7}$$

We also have the integral representation for solution of (1.4) as :

$$\begin{aligned} y(t) &= x(t) + \int_{t_0}^{t} \Phi(s,t_0,v(s))\Phi^{-1}(s,t_0,v(s))[G(s,s,y(s), \\ &\int_{t_0}^{s} h(s,\tau,y(\tau))dt) + \int_{t_0}^{s} G_s(s,\tau,y(\tau),\int_{t_0}^{t} h(\tau,n,y(n))dn)d\tau]ds. \end{aligned} \tag{1.8}$$

Proof. Let $x(t,t_0,x_0)$ be any solution of

$$\begin{aligned} \dot{x}(t) &= \dot{f}(t) + H(t,t,x(t),\int_{t_0}^{t} k(t,\tau,x(\tau))d\tau) \\ &\quad + \int_{t_0}^{t} H_t(t,s,x(s),\int_{t_0}^{s} k(s,\tau,x(\tau))d\tau)ds, x(t_0) = f(t_0) \end{aligned} \tag{1.9}$$

existing for $t \geq t_0$. Determining a function $v(t)$ so that :

$$y(t) = x(t,t_0,v(t)), v(t_0) = x_0 \tag{1.10}$$

is a solution of :

$$\begin{aligned} \dot{y}(t) &= \dot{f}(t) + H(t,t,y(t),\int_{t_0}^{t} K(t,\tau,y(\tau))d\tau) \\ &\quad + \int_{t_0}^{t} H_t(t,s,y(s),\int_{t_0}^{s} k(s,\tau,y(\tau))d\tau)ds \\ &\quad + G(t,t,y(t),\int_{t_0}^{t} h(t,\tau,y(\tau)d\tau) \\ &\quad + \int_{t_0}^{t} G_t(t,s,y(s),\int_{t_0}^{s} h(s,\tau,y(\tau))d\tau)ds, y(t_0) = f(t_0). \end{aligned} \tag{1.11}$$

Differentiating (1.10) with respect to t and using the definition of $\Phi(t,t_0,x_0)$ we get :

$$\dot{y}(t) = \dot{x}(t,t_0,v(t)) + \Phi(t,t_0,v(t))\dot{v}(t).$$

From (1.11), we obtain

$$\begin{aligned}
\dot{y}(t) &= \dot{f}(t) + H(t,t,x(t,t_0,v(t)), \int_{t_0}^{t} k(t,\tau,x(\tau,t_0,v(\tau))d\tau) \\
&\quad + \int_{t_0}^{t} H_t(t,s,x(s,t_0,v(s)), \int_{t_0}^{s} k(s,\tau,x(\tau,t_0,v(\tau))d\tau)ds \\
&\quad + G(t,t,x(t,t_0,v(t)), \int_{t_0}^{t} h(t,\tau,x(\tau))d\tau) \\
&\quad + \int_{t_0}^{t} G_t(t,s,x(s,t_0,v(s)), \int_{t_0}^{s} h(s,\tau,x(\tau,t_0,v(\tau))d\tau)ds \\
&= \dot{x}(t) + \Phi(t,t_0,v(t))\dot{v}(t).
\end{aligned}$$

$$\begin{aligned}
\Phi(t,t_0,v(s))\dot{v}(t) &= G(t,t,y(t), \int_{t_0}^{t} h(t,\tau,y(\tau))d\tau \\
&\quad + \int_{t_0}^{t} G_t(t,s,y(s), \int_{t_0}^{s} h(s,\tau,y(\tau))dr)ds.
\end{aligned}$$

Then

$$\begin{aligned}
\dot{v}(t) &= \Phi^{-1}(t,t_0,v(s))[G(t,t,y(t), \int_{t_0}^{t} h(t,\tau,y(\tau))d\tau) \\
&\quad + \int_{t_0}^{t} G_t(t,s,y(s), \int_{t_0}^{s} h(s,\tau,y(\tau))d\tau ds]. \qquad (1.12)
\end{aligned}$$

Solution of (1.12) then determine $v(t)$. Consequently, if $v(t)$ is a solution of (1.12) then $y(t)$ given by (1.10) is a solution of (1.11). From (1.12), $v(t)$ must satisfy the integral equation

$$\begin{aligned}
v(t) &= f(t_0) + \int_{t_0}^{t} \Phi^{-1}(s,t_0,v(s))[G(s,s,y(s), \int_{t_0}^{s} h(s,\tau,y(\tau))d\tau) \\
&\quad + \int_{t_0}^{s} G_s(s,\tau,y(\tau), \int_{t_0}^{t} h(\tau,u,y(u))du)d\tau]ds,
\end{aligned}$$

which establishes (1.5). Notice that $y(t)$ exists for those values of $t \geq t_0$ for which $v(t)$ exists. From (1.7) we get :

$$x(t,t_0,v(t)) = x(t,t_0,x_0) + \int_{t_0}^{t} \Phi(t,t_0,v(s))\dot{v}(s)ds$$

since $y(t) = x(t, t_0, v(t))$.

$$\begin{aligned}\dot{v}(t) &= \Phi^{-1}(t, t_0, v(s))[G(t, t, y(t), \int_{t_0}^{t} h(t, \tau, y(\tau))d\tau) \\ &\quad + \int_{t_0}^{t} G_t(t, s, y(s), \int_{t_0}^{s} h(s, \tau, y(\tau))d\tau)ds].\end{aligned}$$

Then

$$\begin{aligned}y(t) &= x(t) + \int_{t_0}^{t} \Phi(s, t_0, v(s))\Phi^{-1}(s, t_0, v(s))[G(s, s, y(s), \\ &\quad \cdot \int_{t_0}^{s} h(s, \tau, y(\tau))d\tau) + \int_{t_0}^{s} G_s(s, \tau, y(\tau), \int_{t_0}^{t} h(\tau, u, y(u))du)d\tau]ds.\end{aligned}$$

This completes the proof.

Theorem 1.2. *Suppose that the system (1.1) admits unique solutions* $x(t, t_0, x_0)$. *Suppose, also, that* $\Phi(t, t_0, x_0) \equiv \frac{\partial x}{\partial x_0}(t, t_0, x_0)$ *exists and is continuous for all* $t \geq t_0$, *and that* $\Phi^{-1}(t, t_0, x_0)$ *exists for all* $t \geq t_0$. *If* $v(t)$ *is a solution of :*

$$\dot{v}(t) = \Phi^{-1}(t, t_0, v(t))g(t, y(t), \int_{t_0}^{t} h(t, s, y(s))ds, v(t_0) = x_0 \qquad (1.13)$$

then any solution $y(t, t_0, x_0)$ *of (1.2) satisfies the relation :*

$$\begin{aligned}y(t, t_0, x_0) &= x(t, t_0, x_0 + \int_{t_0}^{t} \Phi^{-1}(s, t_0, v(s))g(s, y(s, t_0, x_0), \qquad (1.14) \\ &\quad \int_{t_0}^{s} h(s, \tau, y(\tau, t_0, x_0))d\tau)ds).\end{aligned}$$

as far as $v(t)$ *exists to the right of* t_0.

Proof. The proof of this theorem follows by the similar argument as in the proof of Theorem 1.1, with suitable modifications and hence we omit the details.

Theorem 1.3. *Under the assumption of Theorem 1.2, the following relation is also valid :*

$$\begin{aligned}y(t, t_0, x_0) &= x(t, t_0, x_0) + \int_{t_0}^{t} \Phi(t, t_0, v(s))\Phi^{-1}(s, t_0, v(s)) \\ &\quad \times g(s, y(s, t_0, x_0), \int_{t_0}^{s} h(s, \tau, y(\tau, t_0, x_0))dr)ds.\end{aligned}$$

Proof. For $t_0 \leq s \leq t$ we have

$$\frac{d}{ds}x(t,t_0,v(s)) = \Phi(t,t_0,v(s))\dot{v}(s).$$

Integrating from t_0 to t we get :

$$x(t,t_0,v(t)) = x(t,t_0,x_0) + \int_{t_0}^{t} \Phi(t,t_0,v(s))\dot{v}(s)ds.$$

Let $v(t)$ be a solution of :

$$\dot{v}(t) = \Phi^{-1}(t,t_0,v(t))g(t,y(t),\int_{t_0}^{t} h(t,s,y(s))ds, v(t_0) = x_0.$$

Since $y(t,t_0,x_0) = x(t,t_0,v(t)), v(t_0) = x_0$. Then

$$\begin{aligned} y(t,t_0,x_0) &= x(t,t_0,x_0) + \int_{t_0}^{t} \Phi(t,t_0,v(s))\Phi^{-1}(s,t_0,v(s)) \\ &\quad g(s,y(s,t_0,x_0),\int_{t_0}^{s} h(s,\tau,y(\tau,t_0,x_0)d\tau)ds. \end{aligned}$$

Before proceeding further, we present a representation formula relating the solutions of (1.3) and its perturbed system (1.4), where $t_0 \in I$, $f \in C[I,R^n]$, $k,h \in C[I\times I\times R^n, R^n]$, and $H,G,H_t,G_t \in C[I\times I\times R^n\times R^n, R^n]$. The integral equation (1.3) is equivalent to

$$\begin{aligned} \dot{x}(t) &= \dot{f}(t) + H(t,t,x(t),\int_{t_0}^{t} k(t,\tau,x(\tau))d\tau) \\ &\quad + \int_{t_0}^{t} H_t(t,s,x(s),\int_{t_0}^{s} k(s,\tau,x(\tau))d\tau)ds, x(t_0) = f(t_0), \end{aligned} \tag{1.15}$$

and (1.4) is equivalent to

$$\begin{aligned} \dot{y}(t) &= \dot{f}(t) + H(t,t,y(t),\int_{t_0}^{t} k(t,\tau,y(\tau))d\tau) \\ &\quad + \int_{t_0}^{t} H_t(t,s,y(s),\int_{t_0}^{s} k(s,\tau,y(\tau))d\tau)ds \\ &\quad + G(t,t,y(t),\int_{t_0}^{t} h(t,\tau,y(\tau))d\tau) \\ &\quad + \int_{t_0}^{t} G_t(t,s,y(s),\int_{t_0}^{s} h(s,\tau,y(\tau))d\tau)ds, y(t_0) = f(t_0). \end{aligned} \tag{1.16}$$

Now solution of (1.15) are dependent on the initial data and we write $x(t; t_0, x_0)$ to denote solution of (1.15) and note that the trajectory $x(t; t_0, f(t_0))$ produces a solution of the integral equation (1.3). As in Theorem 1.2, we adapt the notation

$\Phi(t, t_0, x_0) \equiv \frac{\partial x}{\partial x_0}(t, t_0, x_0)$. This completes the proof.

In this section we shall use our representation formula (1.7) and the integral inequality given in Lemma 1.1, to study the stablility, boundedness, and asymptotic behaviour of the solutions of perturbed integrodifferential equation 3.1.2, under some suitable conditions on the solutions of (1.1) and (1.13) and the perturbation terms involved in (1.2).

Theorem 1.4. *Let the solution $x(t)$ of (1.1) be uniformly slowly growing. Assume the hypothesis of Theorem 1.3, hold, and the solution $v(t)$ of (3.1.13) satisfies the condition:*

$$|\Phi(t, t_0, v(s))\Phi^{-1}(s, t_0, v(s))| \leq Ne^{-\alpha(t-s)}, 0 \leq s \leq t < \infty \tag{1.17}$$

where N and α are positive constants, suppose that g and h in (1.2) satisfy

$$|g(t, y, u)| \leq p(t)[|y| + |u|], t \in I \tag{1.18}$$

$$|h(t, s, y)| \leq e^{\alpha(t-s)}q(s)|y|, 0 \leq s \leq t < \infty \tag{1.19}$$

where $p, q \in C[I, R^+]$ and $\int_{t_0}^{\infty} p(s)ds < \infty, \int_{t_0}^{\infty} q(s)ds < \infty$. Then all solutions of (1.2) are slowly growing.

Proof. Using (1.17); (1.18), and (1.19) in

$$\begin{aligned} y(t) &= x(t) + \int_{t_0}^{t} \Phi(t, t_0, v(s))\Phi^{-1}(s, t_0, v(s))g(s, y(s), \\ &\qquad \int_{t_0}^{s} h(s, \tau, y(\tau))d\tau)ds \end{aligned}$$

together with uniformly slowly growing.

$$\begin{aligned} |y(t)| &\leq M|x_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Ne^{-\alpha(t-s)}p(s)[|y(s)| \\ &\qquad + e^{-\alpha s}\int_{t_0}^{s} q(\tau)|y(\tau)|e^{\alpha\tau}dt]ds. \end{aligned}$$

The above inequality can be rewritten as :

$$\begin{aligned} |y(t)|e^{\alpha t} &\leq M|x_0|e^{\alpha t_0} + \int_{t_0}^{t} Np(s)[|y(s)|e^{\alpha s} \\ &\qquad + \int_{t_0}^{s} q(\tau)|y(t)|e^{\alpha t}d\tau]ds. \end{aligned}$$

Now applying lemma (1.1), in which $u(t)$ is replaced by $|y(t)|e^{\alpha t}$ then multiplying by $e^{-\alpha t}$, we get :

$$|y(t)| \leq M|x_0|e^{-\alpha(t-t_0)}[1 + \int_{t_0}^{t} Np(s)\exp(\int_{t_0}^{s}[Np(\tau)+q(\tau)]d\tau)ds].$$

$$|y(t)| \leq M|x_0|e^{-\alpha(t-t_0)}[1 + \int_{t_0}^{s} Np(s)\exp(\int_{t_0}^{s}[Np(\tau)+q(\tau)]d\tau)ds].$$

Let $K = \int_{t_0}^{t} Np(s)\exp(\int_{t_0}^{s}[Np(\tau)+q(\tau)]d\tau)ds$.
Then

$$\int_{t_0}^{s} Np(s)\exp(\int_{t_0}^{s}[Np(\tau)+q(\tau)]d\tau ds \leq k$$

$$|y(t)| < M|x_0|e^{-\alpha(t-t_0)}[1+k].$$

The above estimate yields the desired result if we choose M and $|x_0|$ small enough. This completes the proof.

Theorem 1.5. *Let the solution $x(t)$ of (1.1) be globally uniformly stable. Assume the hypothesis of Theorem (1.3), holds, and the solution $v(t)$ of (1.13) satisfies the condition*

$$|\Phi(t,t_0,v(s))\Phi^{-1}(s,t_0,v(s))| \leq N, 0 \leq s \leq t < \infty \tag{1.20}$$

where $N > 0$ is a constant. Suppose that g, and h in (3.1.2) satisfy

$$|g(t,y,u)| \leq p(t)[|y|+|u|], t \in I \tag{1.21}$$

$$|h(t,s,y)| \leq q(s)|y|, \quad 0 \leq s \leq t < \infty \tag{1.22}$$

where $p, q \in C[I, R^+]$ and

$\int_{t_0}^{\infty} p(s)ds < \infty$ and $\int_{t_0}^{\infty} q(s)ds < \infty$. Then all solution of (1.2) are bounded on I.

Theorem 1.6. *Let the solution $x(t)$ of (1.1) be exponentially asymptically stable. Assume the hypothesis Theorem 3.1.3, holds, and the solution $v(t)$ of (1.13) satisfies the condition*

$$|\Phi(t,t_0,v(s))\Phi^{-1}(s,t_0,v(s))| \leq Ne^{-\alpha(t-s)}, \quad 0 \leq s \leq t < \infty \tag{1.23}$$

where N and α are positive constants. Suppose that g and h in (1.2) satisfy

$$|g(t,y,u)| \leq p(t)[|y|+|u|], \quad t \in I \tag{1.24}$$

$$|h(t,s,y)| \leq e^{\alpha(t-s)}q(s)|y|, \quad 0 \leq s \leq t < \infty \tag{1.25}$$

where $p, q \in C[I, R^+]$ *and* $\int_{t_0}^{\infty} p(s)ds < \infty, \int_{t_0}^{\infty} q(s)ds < \infty$. *Then all solutions of (1.2) approach zero as* $t \to \infty$.

The proof of Theorems 1.5 and 1.6, follows by the similar argument as in the proof of Theorem 1.4, with suitable modifications and hence we omit the details.

We remark that our result in this section can be modified very easily to the study of behaviour of solutions of (1.3) and (1.4) by using the modified version of lemma 1.1, under some suitable conditions on the functions involved in (1.4).

In concluding this section, we note that although the result on the stability of the solutions of some special forms of (1.1) and (1.2) have been considered in a recent section by Bernfeld and Lord [2], the results presented here are of interest because of the weak assumptions on the functions involved and the approach to the problem is different from those of Bernfeld and Lord [2].

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MATHEMATICS DEPARTMENT, FACULTY OF SCIENCE, SANA'A UNIVERSITY, SANA'A, P.O. BOX 1247, YEMEN ARAB REPUBLIC.