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A Note on Separable Algebras

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I. Introduction

Let A be an R-algebra, and let M be a two sided A/R-module. For an element $g \in \mathrm{Hom}_R(A, M)$ if it satisfies the condition

$$g(ab)=ag(b)+g(a)b$$

for all a, $b \in A$, then g is said to be a ***derivation***. In particular if there exists an element $m \in M$ such that

$$g(a)=am-ma(a \in A)$$

then g is called an ***inner derivation***.

It is clear that an inner derivation g is a derivation because that for all a, $b \in A$

$$g(ab)=abm-mab,$$

and on the other hand

$$\begin{aligned} ag(b)+g(a)b &= a(bm-mb)+(am-ma)b \\ &= abm-amb+amb-mab \\ &= abm-mab, \end{aligned}$$

where m is an element of M.

Moreover, if $g \in \mathrm{Hom}_R(A, M)$ is derivation then $g(1)=0$, because that $g(1 \cdot 1)=g(1)=g(1)+g(1)$ implies that $g(1)=0$.

Let us put

$$Z_R^1(A, M)=\{f \in \mathrm{Hom}_R(A, M) \mid f \text{ is a derivation}\}$$

$$B_R^1(A, M)=\{f \in Z_R^1(A, M) \mid f \text{ is an inner derivation}\}.$$

Then it is obvious that

1^0. $Z^1_R(A, M)$ and $B^1_R(A, M)$ are R-modules.

2^0. $B^1_R(A, M)$ is an R-submodule of $Z^1_R(A, M)$.

Definition 1.1. With the above notations

$$H^1_R(A, M)=Z^1_R(A, M)/B^1_R(A, M),$$

which is an R-module, is called the *first Hochschild cohomology module of A with coefficients in M*([5]).

The purpose of this paper is to show that an R-algebra A is R-separable if and only if $H^1_R(A, M)=0$ for all two sided A/R-module M.

II. Preliminaries

Throughout this paper, we shall assume that R is a commutative ring with 1. For an R-algebra A, A^0 denotes the R-algebra opposite to A.
We shall set

$$A^e=A\otimes_R A^0$$

and call it the *enveloping algebra of A*. In this case, A has a structure as a left A^e-module induced by

$$(a\otimes a')b=aba', \quad (a\otimes a'\in A^e,\ b\in A).$$

We define an A^e-module homomorphism

$$\mu:\ A\otimes A^0 \longrightarrow A$$

defined by $\mu(\sum_i a_i\otimes a_i')=\sum_i a_i\cdot a_i'$.
It follows that if A is commutative then μ is a ring homomorphism. We put Ker $\mu=J$, then we have an exact sequence of left A^e-modules :

$$0\to J\to A^e \xrightarrow{\mu} A\to 0 \qquad (*)$$

Proposition 2.1. *J is the left ideal of A^e generated by all elements of the form*

$a\otimes 1-1\otimes a,\ a\in A.$

Proof. It is obvious that

$$\mu(a\otimes 1-1\otimes a)=a-a=0.$$

Suppose that

$$\mu(a\otimes a')=aa'=0.$$

Then

$$(a\otimes 1)(1\otimes a'-a'\otimes 1)=a\otimes a'.$$

Therefore, $(A\otimes A^0)\{a\otimes 1-1\otimes a \mid a\in A\}=J.$ / / /

We have the following ([1], [2], [3], [4]).

Property 2°. *The following conditions on an R-algebra A are equivalent :*

(i) *A is an A^e-projective module under the μ-structure.*

(ii) *The above sequence (*) splits as a sequence of left A^e-modules.*

(iii) *A^e contains an element e, which is called a separability idempotent, such that $\mu(e)=1$ and $Je=0$.*

Definition 2.2. An R-algebra A is said to be *separable* if it satisfies the equivalent conditions of Property 2^0.

Theorem 2.3. *Let A be a separable R-algebra, and let*

$$0 \to L \to M \xrightarrow{\eta} N \to 0 \quad (**)$$

be an exact sequence of A-modules. If (**) splits as a sequence of R-modules then (**) also splits as an exact sequence of A-modules.

Proof. By our hypothesis there exists an R-module homomorphism $\Psi : N \to M$ such that $\eta\circ\Psi=1_N$. We put

$$\Psi'=e\cdot\Psi : N \to M,$$

where $e=\sum_i x_i\otimes y_i$ is a separability idempotent of A. We have to note that $\mathrm{Hom}_R(N, M)$ is a two sided A/R-module (i.e., a left A^e-module) with operations

$(a\otimes a')\cdot\Psi=a\Psi a'$,

that is,

$$((a\otimes a')\cdot\Psi)(n)=a\Psi(a'n)$$

for any $n\in N$, where $a\otimes a'\in A^e$ and $\Psi\in \mathrm{Hom}_R(N, M)$. In this case, for all $n\in N$

$\eta\circ\Psi'(n)=n$ because that

$$\begin{aligned}\eta\circ\Psi'(n)&=\eta(e\Psi(n))=\eta(\textstyle\sum_i x_i\otimes y_i)\Psi(n)\\&=\eta(\textstyle\sum_i x_i\Psi y_i)(n)\\&=\eta(\textstyle\sum_i x_i\Psi(y_in))\\&=\textstyle\sum_i x_i\eta\circ\Psi(y_in)\\&=\textstyle\sum_i x_iy_in\\&=n\end{aligned}$$

(Note that $\sum_i x_iy_i=1$).

Moreover, Ψ' is an A-module homomorphism. In fact, since for all $a\in A$ $(a\otimes 1 - 1\otimes a)e=0$ we have

$$(a\otimes 1)e\Psi=(1\otimes a)e\Psi \quad (a\in A).$$

Thus for all $a\in A$ and for all $n\in N$

$$\begin{aligned}a\Psi'(n)&=((a\otimes 1)e\Psi)(n)=((1\otimes a)e\Psi)(n)\\&=(\textstyle\sum_i x_i\otimes y_ia\cdot\Psi)(n)\\&=\textstyle\sum_i x_i\Psi(y_ian)\\&=(\textstyle\sum_i (x_i\otimes y_i)\cdot\Psi)(an)\\&=e\cdot\Psi(an)\\&=\Psi'(an).\end{aligned}$$

Therefore the above (**) splits as a sequence of A-modules. / / /

Ⅲ. Main Result

We shall prove two properties with respect to separability.

Theorem 3.1. *A is a separable R-algebra if and only if $H_R^1(A, M)=0$ for all two-sided A/R-module M.*

Proof. ⇨ : Let A be R-separable. Then there exists a separability idempotent $e=\sum_i x_i\otimes y_i$ (see Property 2^0). As in the proof of Theorem 2.3, since $\mathrm{Hom}_R(A, M)$ is a two sided A/R-module, for each $g\in Z_R^1(A, M)$

$$((a\otimes 1-1\otimes a)e\cdot g)(1)=0$$
$$\Rightarrow ((a\otimes 1)\cdot eg)(1)=((1\otimes a)\cdot eg)(1)$$

and thus for all $a\in A$

$$\begin{aligned}(\textstyle\sum_i ax_i\otimes y_i\cdot g)(1)&=(\textstyle\sum_i x_i\otimes y_ia\cdot g)(1)\\ \textstyle\sum_i ax_ig(y_i) &= \textstyle\sum_i x_ig(y_ia)=\sum_i x_ig(y_i)a+\sum_i x_iy_ig(a)\\ &=\textstyle\sum_i x_ig(y_i)a+g(a)\end{aligned}$$

$(\sum_i x_iy_i=1)$.
Hence

$$g(a)=\sum_i ax_ig(y_i)-\sum_i x_ig(y_i)a.$$

We put $m=\sum_i x_ig(y_i)$ then for all $a\in A$

$$g(a)=am-ma.$$

That is, $g\in Z_R^1(A, M)\Rightarrow g\in B_R^1(A, M)$. We have $H_R^1(A, M)=0$.

⇦ : For all two sided A/R-module M we assume $H_R^1(A, M)=0$. Recall that $J=\mathrm{Ker}\mu$ is a two-sided A/R-module.
Hence, by our assumption $H_R^1(A, J)=0$. Take $[\tau]\in H_R^1(A, J)$ such that

$$\text{for all } a\in A \quad \tau(a)=a\otimes 1-1\otimes a.$$

This is well-defined, because that for all $a, b\in A$

$$\begin{aligned}\tau(ab)&=a\tau(b)+\tau(a)b\\ &=a(b\otimes 1-1\otimes b)+(a\otimes 1-1\otimes a)b\\ &=ab\otimes 1-a\otimes b+a\otimes b-1\otimes ab\\ &=ab\otimes 1-1\otimes ab.\end{aligned}$$

Since $H^1_R(A, J)=0$, we have $\tau \in B^1_R(A, J)$, and there exists an element $m \in J$ such that for all $a \in A$

$$\tau(a)=am-ma.$$

Thus we have the following :

$$am-ma=a\otimes 1-1\otimes a$$

and so

$$(a\otimes 1)(1\otimes 1-m)-(1\otimes a)(1\otimes 1-m)=0.$$

Therefore, $J(1\otimes 1-m)=0$. Moreover $\mu(1\otimes 1-m)=1$ because that $\mu(m)=0$. This means that $1\otimes 1-m$ is a separability idempotent of A. Therefore, by Property 2^0, A is a separable R-algebra. / / /

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