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On the Indices and the Bordism Groups*

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Let MSO_m be the m-dimensional Thom's bordism groups ([2]), and let $\boldsymbol{Q}$ be the field of rationals. For $\{M^{4n}\} \in MSO_{4n}$ $H^{2n}(M^{4n};\boldsymbol{Q})$ is a free group ([2]), and we assume $\{\alpha_1, \cdots, \alpha_r\}$ is a basis of $H^{2n}(M^{4n};\boldsymbol{Q})$. Suppose the $r\times r$-matrix

$$A=(<\alpha_i \cup \alpha_j, [M^{4n}]>) \ (1\leqq i, j\leqq r),$$

where $<, >$ is a kronecher product, $[M^{4n}]$ is the fundamental homology class of M^{4n} and $\cup$ is the cup product. Then

$$|<\alpha_i \cup \alpha_j, [M^{4n}]>|=\pm 1, \ (1\leqq i, j\leqq r)$$

([2]). We define the index $I[M^m]$ of $\{M^m\} \in MSO_m$ such that

(a) $m \not\equiv 0 \bmod 4 \Rightarrow I[M^m]=0$

(b) $m=4n \Rightarrow I[M^{4n}]=p-q$ where p and q are defined as follows:

We define the quadratic form

$$\begin{aligned}\boldsymbol{Q}(\alpha)=\boldsymbol{Q}(\alpha_1, \cdots, \alpha_r)&=\sum_{1\leqq i,j\leqq r}<\alpha_i \cup \alpha_j, [M^{4n}]>\alpha_i\alpha_j\\ &=\sum_{1\leqq i,j\leqq r} a_{ij}\alpha_i\alpha_j\end{aligned}$$

Then $\boldsymbol{Q}(\alpha)$ is equivalent to the quadratic form

$$\sum_{i=1}^{p}\alpha_i^2-\sum_{j=1}^{q}\alpha_{p+j}^2$$

where $p+q$ is the rank of the matrix A which is the discriminant of $\boldsymbol{Q}(\alpha)$.

Let $L[M^{4n}]$ be the L-genus of M^{4n}(see Definition 2). The purpose of this note is to introduce the first stage of the Atiyah-Singer index theorem and to prove that

(c) $\{M^{4n}\}=0$ in $MSO_{4n} \Rightarrow I(M^{4n})=0$

(d) $I(M^{4n})=L[M^{4n}]$ for $\{M^{4n}\} \in MSO_{4n}$.

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Let A^0, A^1, $\cdots$, A^n, $\cdots$, be $\boldsymbol{Q}$ −modules satisfying

$$A^i \cdot A^j \subset A^{i+j} \ (i,\ j=0,\ 1,\ 2,\ \cdots).$$

Let us put

$$A^* = \oplus A^i \text{ (direct sum)},$$

then it is a graded algebra over $\boldsymbol{Q}$. We also put

$$A_0^{\Pi} = \{a_0 + a_1 + a_2 + \cdots \mid a_i \in A^i,\ a_0 = 1 \in A^0\},$$

then for x, $y \in A_0^{\Pi}$ it is clear that $xy \in A_0^{\Pi}$.

Let us consider a sequence of polynomials with coefficients in $\boldsymbol{Q}$;

$$K_1(x_1),\ K_2(x_1,\ x_2),\ \cdots,\ K_n(x_1,\ \cdots,\ x_n),\ \cdots$$

such that $K_n(x_1, \cdots, x_n)$ is a homogeneous polynomial of degree $n(n=1, 2, \cdots)$ of degree of $x_i = i$ for $i=1, 2, \cdots$.

For each $a = 1 + a_1 + a_2 + \cdots \in A_0^{\Pi}$ we define such as

$$K(a) = 1 + K_1(a_1) + K_2(a_1,\ a_2) + \cdots + K_n(a_1,\ \cdots,\ a_n) + \cdots.$$

If $K(ab) = K(a)K(b)$ for each a, $b \in A_0^{\Pi}$, then the sequence $\{K_n(a_1, \cdots, a_n)\}$ $n=1, 2, \cdots$ of polynomials is called a ***multiplicative sequence over*** $\boldsymbol{Q}$.

We have the following lemma in [2] :

Lemma 1. For $\boldsymbol{Q}[t] = A^*$ $(A^0 = \boldsymbol{Q},\ A^1 = \boldsymbol{Q}t,\ \cdots)$ (degree of $t=1$) and for

$$f(t) = 1 + \lambda_1 t + \lambda_2 t^2 + \cdots \in A_0^{\pi} \ (\lambda_i \in \boldsymbol{Q})$$

there exists a unique multiplicative sequence $\{K_n\}$ such that

$$f(t) = K(1+t) = 1 + K_1(t) + K_2(t,\ 0) + \cdots$$

(Note that $1+t = 1+t+0+0+ \cdots \in A_0^{\pi}$.)

Let $\{K_n(x_1, \cdots, x_n)\}$ be a multiplicative sequence over $\boldsymbol{Q}$.

For $\{M^m\} \in MSO_m$ we define $K[M^m] \in \boldsymbol{Q}$ as follows :

(i) $m \not\equiv 0 \bmod 4 \Rightarrow K[M^m] = 0$

(ii) $m=4n \Rightarrow$ for the Pontrjagin class $P_i(M^{4n})$ of M^{4n}

$$K[M^{4n}]=<K_n(P_1(M^{4n}), \cdots, p_n(M^{4n})), [M^{4n}]>.$$

In this case $K[M^{4n}] \in \boldsymbol{Q}$ is the K-genus of M^{4n}.

Then $K[M^{4n}]$ is a linearly composition of Pontrjagin numbers over $\boldsymbol{Q}$ ([1], [2]).

Consider the hyperbolic tangent

$$\tanh x=\frac{e^x-e^{-x}}{e^x+e^{-x}}.$$

Then

$$\sqrt{t}\,/\tanh\sqrt{t}=1+\frac{1}{3}t-\frac{1}{45}t^2+\cdots+\frac{(-1)^{k-1}2^{2k}B_k}{(2k)!}t^k+\cdots=f(t)$$

where B_k is a Bernoulli number such that

$$B_1=1/6, \quad B_2=1/30, \quad B_3=1/42, \cdots.$$

Then by the above Lemma 1 there exists a unique multiplicative sequence such that

$$L(1+t)=f(t)=\sqrt{t}\,/\tanh\sqrt{t}.$$

Let $p_i(M^{4n})$ $(i=1, \cdots, n)$ be the Pontrjagin class of M^{4n} where $\{M^{4n}\} \in MSO_{4n}$. Then $L_m(p_1(M^{4n}), \cdots, p_m(M^{4n}))$ $(m=1, 2, \cdots, n)$ is the following

$$L_1(p_1(M^{4n}))=1/3 \cdot p_1(M^{4n})$$

$$L_2(p_2(M^{4n}), p_n(M^{4n}))=1/45(7p_2(M^{4n})-(p_1(M^{4n}))^2)$$

....

$$L_n(p_1(M^{4n}), \cdots, p_n(M^{4n}))=2^n(2^{2n-1}-B_n)/(2n)!p_n(M^{4n})$$

$$+\text{the terms containing of } p_1(M^{4n}), \cdots, \text{ and } p_{n-1}(M^{4n}).$$

Definition 2. (L-genus). For $\{M^m\} \in MSO_m$ the L-genus $L[M^m]$ of M^m is defined by

(i) $m \not\equiv 0 \bmod 4 \Rightarrow L[M^m]=0$

(ii) $m=4n \Rightarrow L[M^{4n}]=<L_n(p_1(M^{4n}), \cdots, p_n(M^{4n})), [M^{4n}]>$.

It is well-known that the bordism groups are deeply related to the index theory ([2], [3]). The following lemma is one of these relations.

Lemma 3. If $\{M^{4n}\}=0$ in MSO_{4n} then $I[M^{4n}]=0$.

Proof. Since $\{M^{4n}\}=0$ in MSO_{4n} there exists an oriented compact 4n+1-dimensional real C^{∞} manifold W^{4n+1} such that ∂W^{4n+1} (the boundary of W^{4n+1})$=M^{4n}$.
By the Poincaré-Lefeschetz dual theorem([1], [2]) we have the commutative diagram(∗), such that

$$\begin{array}{ccccccccc}
\cdots\to H^q(W^{4n+1}) & \xrightarrow{i^*} & H^q(M^{4n}) & \xrightarrow{\delta^*} & H^{q+1}(W^{4n+1}, M^{4n}) & \longrightarrow & H^{q+1}(W^{4n+1}) & \longrightarrow\cdots(\text{exact}) & \\
\downarrow \cap[W^{4n+1}, M^{4n}] \ \copyright & & \downarrow \cap[M^{4n}] \ \copyright & & \downarrow \cap[W^{4n+1}, M^{4n}] \ \copyright & & \downarrow \cap[W^{4n+1}, M^{4n}] & & (*) \\
\cdots\to H_{4n+1-q}(W^{4n+1}, M^{4n}) & \xrightarrow{\partial_*} & H_{4n-q}(M^{4n}) & \xrightarrow{i_*} & H_{4n-q}(W^{4n+1}) & \longrightarrow & H_{4n-q}(W^{4n+1}, M^{4n}) & \to\cdots(\text{exact}) &
\end{array}$$

where $\cap$ is the cap product and $H^q(W^{4n+1})=H^q(W^{4n+1}; \boldsymbol{Q})$, and so on.
We put

$$A^q=Im\ i^*\subset H^q(M^{4n})$$
$$K_{4n-q}=Ker(i_*: H_{4n-q}(M^{4n}) \to H_{4n-q}(W^{4n+1})).$$

By the exactness of the cohomology and homology long exact sequences, we have

$$\cap[M^{4n}] : A^q \xrightarrow{\cong} K_{4n-q} .$$

For each $\alpha\in A^q$ and $\beta\in A^{4n-q}$ there exist $\alpha'\in H^q(W^{4n+1})$ and $\beta'\in H^{4n-q}(W^{4n+1})$ such that

$$i^*(\alpha')=\alpha, \quad i^*(\beta')=\beta.$$

Then

$$\begin{aligned}
<\alpha\cup\beta, [M^{4n}]> &= <i^*(\alpha')\cup i^*(\beta'), \partial_*([W^{4n+1}, M^{4n}])\\
&= <\delta^* i^*(\alpha')\cup\delta^* i^*(\beta'), [W^{4n+1}, M^{4n}]>\\
&= 0.
\end{aligned}$$

Since $H^i(M^{4n}; \boldsymbol{Q})\cong H_i(M^{4n}; \boldsymbol{Q})$ and $H^i(W^{4n+1}; \boldsymbol{Q}\cong H_i(W^{4n+1}; \boldsymbol{Q})$ are free groups the commutative diagram

$$\begin{array}{ccc}
H^{4n-q}(W^{4n+1}) & \xrightarrow{\quad i^* \quad} & H^{4n-q}(M^{4n}) \\
\downarrow\cong & & \downarrow\cong \\
\mathrm{Hom}(H_{4n-q}(W^{4n+1}), \boldsymbol{Q}) & \xrightarrow{\quad h(i^*) \quad} & \mathrm{Hom}(H_{4n-q}(M^{4n}), \boldsymbol{Q})
\end{array}$$

where for $f\in \mathrm{Hom}(H_{4n-q}(W^{4n+1}), \boldsymbol{Q})$ $h(i_*)(f)=f\circ i_*$, says the following :

$$H^{4n-q}(M^{4n}) \cong A^{4n-q} \oplus Hom(K_{4n-q}, \mathbf{Q})$$

if we identify $H^{4n-q}(M^{4n})$ and $\mathrm{Hom}(H_{4n-q}(M^{4n}), \mathbf{Q})$.

For $\alpha \in A^q$ $\alpha \cap [M^{4n}] \in K_{4n-q}$ because that in the diagram (∗) above $i_*(\alpha \cap [M^{4n}])=0$ (note that $\delta^* \alpha=0$). Therefore we can take $\alpha \in A^q$ and $\alpha' \in \mathrm{Hom}(K_{4n-q}, \mathbf{Q})$ such that $<\alpha' \cup \alpha, [M^{4n}]> =0$. When q=2n, since

$$H^{2n}(M^{4n}; \mathbf{Q}) \cong A^{2n} \oplus \mathrm{Hom}(K_{2n}, \mathbf{Q})$$

we can take basis $\{\alpha_1, \cdots, \alpha_r\}$ of A^{2n} and $\{\alpha_1', \cdots, \alpha_r'\}$ of $\mathrm{Hom}(K_{2n}, \mathbf{Q})$ such that

$$<\alpha_i' \cup \alpha_j, [M^{4n}]>=0$$

for 1≦i, j≦r. Thus the $2r \times 2r$-matrix which decides the index $I[M^{4n}]$ is equivalent to the following form

$$\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$$ (where E is the unit matrix).

Hence it is clear that $I[M^{4n}]=0$ ([2]). / / /

We shall prove that there exists an elliptic differential operator D_0 on $P^{2k}(\mathbf{C})$ such that

$$i_t(D_0)=(2k+1)L[P^{2k}(\mathbf{C})]=(2k+1)I[P^{2k}(\mathbf{C})]$$

in the next note (to appear), where $i_t(D_0)$ is the ***topological index*** of D_0.
In this note we shall prove the following :

Theorem 4. For each $\{M^{4n}\}$ in MSO_{4n}

$$I[M^{4n}]=L[M^{4n}].$$

Proof. Let $p^{2n}(\mathbf{C})$ be the 2n-dimensional complex projective space. Then

$$H^{2n}(p^{2n}(\mathbf{C}) ; \mathbf{Q}) \cong \mathbf{Q}$$

([1], [2]). Moreover, for the generator $\alpha \in H^2(p^{2n}(\mathbf{C}) ; \mathbf{Q} \cong \mathbf{Q}$, α^n is the generator of

$$H^{2n}(p^{2n}(\mathbf{C}) ; \mathbf{Q}) \cong \mathbf{Q}.$$

Therefore, since

$$<\alpha^n \cup \alpha^n,\ [p^{2n}(C)]>=1,$$

we have that $I[p^{2n}(C)]=1$.

Furthermore, for the total Pontrjagin class $p(p^{2n}(C))$ we have

$$p(p^{2n}(C))=(1+\alpha^2)^{2n+1}.$$

([2]). Therefore, if p_i is a Pontrjagin class then we have the following :

$$L(1+p_1+\ \cdots\ +p_n)=L((1+\alpha^2)^{2n+1})=(L(1+\alpha^2))^{2n+1}$$
$$=(\frac{\alpha}{\tanh\alpha})^{2n+1}$$

(Note that $L(ab)=L(a)L(b)$.) Thus, $L[p^{2n}(C)]$ is the coefficient of α^{2n} in the power series of $(\alpha/\tanh\alpha)^{2n+1}$.

If C is a closed simple curve containing the origin in the complex plane then

$$\int_c z^n dz=\begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{if } n \neq -1 \end{cases}$$

Hence we have the following :

$$L[p^{2n}(C)]=\frac{1}{2\pi i}\int_c \frac{1}{z^{2n+1}}(\frac{z}{\tanh z})^{2n+1}dz.$$

We put $u=\tanh z$ then from

$$\cosh^2 z-\sinh^2 z=1 \text{ and } \tanh z=\sinh z/\cosh z$$

we get

$$1-u^2=1/\cosh^2 z \text{ and } du/dz=1/\cosh^2 z.$$

Moreover, $dz=du/(1-u^2)=(1+u^2+u^4+\ \cdots)du$

and thus

$$L[p^{2n}(C)]=\frac{1}{2\pi i}\int_c \frac{dz}{(\tanh z)^{2n+1}}=\frac{1}{2\pi i}\int_c \frac{1+u^2+\cdots}{u^{2n+1}}du$$
$$=1.$$

That is, we have $I[p^{2n}(C)]=L[p^{2n}(C)]$. Since $MSO_*\otimes \boldsymbol{Q}$ is generated by $\{p^2(C)\}$, $\{p^4(C)\}$, $\cdots$ $\{p^{2n}(C)\}$ $\cdots$ over $\boldsymbol{Q}$ ([2]) our assertion is true. / / /

References

[1] K. Lee and H. Lee : Sheaf theory, Complex Manifolds and Index Theorem, Hyongseol–Sa, Korea (1984).

[2] K. Lee : Foundation of Topology, Hakmoon–Sa, Korea (1981).

[3] R.S. Palais : Seminar on the Atiyah–Singer Index Theorem, Annals of Mathematic Studies No. 57, Princeton Univ. Press (1965).