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# SPACE OF GEOMETRIC STRUCTURES WHOSE DEVELOPMENTS ARE COVERINGS

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## 0. Introduction

Let $X$ be a smooth manifold and $A$ be a Lie group of diffeomorphisms of $X$ with "analytic nature". A smooth manifold $M$ has a $(X, A)$-structure (or an $A$-structure, in short) if it admits an atlas whose coordinate transitions belong to $A$, and in this case, $M$ is said to be locally modelled on $(X, A)$. The examples of $A$-structures in our mind are the classical space forms and various flat structures such as affinely, projectively and conformally flat structures. We can associate the well known model space $X$ and Lie group $A$ appropreately for each of these examples. Yet the notion of $A$-structure is much broader as we can choose, for instance, any Riemannian manifold $X$ and a group of isometries of $X$ as $A$.

For a given $(X, A)$, the fundamental questions are to determine the manifolds admitting $A$-structures and to describe the space of $A$-structures on a given manifold $M$. The deformation space or the moduli space are sometimes studied in conjunction with the representation variety $\mathrm{Hom}(\pi, A)$, $\pi = \pi_1(M)$, via the holonomy representation. This approach is especially useful when the developing map defined on a universal covering $\tilde{M}$ of $M$ is a diffeomorphism onto $X$ so that $A$-structure is complete. In this case, a subset of $A\backslash\mathrm{Hom}(\pi, A)/\mathrm{Aut}(\pi)$ (resp. $A\backslash\mathrm{Hom}(\pi, A)$) faithfully parametrize the $A$-structure on $M$ up to $A$-equivalence (resp. $A$-equivalence homotopic to identity). However if $A$-structure is not complete, in a lot of important cases, the developing map becomes a covering map, and we intend to study the deformation sapce of $A$-structures in this case.

When the developing map $D$ is a covering, $A$ can be lifted to $A_D$, the group of $A$-automorphisms on $\tilde{M}$. Then the canonical projection

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of $A_D$ onto $A$ is a covering homomorphism with the kernel $\Delta$ which is the desk transformation group for $D$. We show in this paper that a subset of $A_D\backslash\mathrm{Hom}(\pi, A_D)/\mathrm{Aut}(\pi)$ (resp. $A\backslash\mathrm{Hom}(\pi, A)$) is in 1-1 correspondence with the moduli space (resp. the deformation space) of $A$-structures whose developing map is a covering, and then show that the canonical projection of $A_D$ onto $A$ induces a covering map from the representation space $\mathrm{Hom}(\pi, A_D)$ into $\mathrm{Hom}(\pi, A)$ with fiber $\mathrm{Hom}(\pi, \Delta)$ under some conditions.

## 1. Basic concepts

In this section, we will set up some basic facts and rudiments for geometric structures for later use. (See also [Go], [Ku], [NY] and [Th] for more about $A$-structures.)

Let $X$ be a model smooth manifold and $A$ be a Lie group consisting of diffeomorphisms of $X$ which are uniquely determined by local data, i.e., for any $a, b \in A$, $a|_U = b|_U$ for a non-empty open set $U \subset X$ implies $a = b$. An *$A$-structure* on a smooth manifold $M$ is a maximal atlas $\{(U_\alpha, \varphi_\alpha)\}$, where $\varphi_\alpha : U_\alpha \to X$ is a smooth coordinate chart, called an $A$-chart, such that $\varphi_\alpha \cdot \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ is a restriction of an element $g_{\alpha\beta}$ of $A$. A manifold with an $A$-structure is called an $A$-manifold. Let $M$ and $N$ be $A$-manifolds. A map $f : M \to N$ is called an *$A$-map* if it is represented locally by an element of $A$, i.e., for any $x \in M$, there are $A$-charts $(U_\alpha, \varphi_\alpha)$ on $M$ and $(V_\beta, \psi_\beta)$ on $N$ containing $x$ and $f(x)$ respectively such that $\varphi_\beta \circ f \circ \varphi_\alpha^{-1}$ is a restriction of an element of $A$. Note that an $A$-map is a local diffeomorphism.

PROPOSITION 1.1. *$A$-map of a connected $A$-manifold $M$ into an $A$-manifold $N$ is uniquely determined by local data, i.e., if $f$, $g$ are $A$-maps : $M \to N$ with $f|_U = g|_U$ for some non-empty open $U \subset M$, then $f = g$.*

*Proof.* Let $W$ be a maximal open set contained in $M$ on which $f$ and $g$ agree. For any $x \in \overline{W}$, choose an $A$-chart $(U_\alpha, \varphi_\alpha)$ at $x$ such that $f$ and $g$ are represented locally as $a, b \in A$ respectively on $\varphi_\alpha(U_\alpha)$. Since $f$ and $g$ agree on a non-empty open set $W \cap U_\alpha$, $a = b$ and hence $f = g$ on $W \cup U_\alpha$. By maximality of $W$, $U_\alpha \subset W$. This shows that $\overline{W} = W$ and hence $W = M$ by connectedness of $M$.

The above proposition says that an $A$-map of a connected manifold

$M$ into $X$ is completely determined by local data if it exists. An $A$-map $D : M \to X$, if exists, is called a *developing* map. In general, a developing map exists on an $A$-manifold $M$, if $M$ is simply connected, by the usual analytic continuation argument. Therefore to develop a given $A$-manifold $M$, we will use the universal covering $\tilde{M}$ of $M$ with the pull-back $A$-structure.

PROPOSITION 1.2. *Let $M$ and $M'$ be connected $A$-manifolds which have developing map $D$ and $D'$ respectively. For any $A$-map $f : M \to M'$, there exists a unique $a \in A$ such that $a \circ D = D' \circ f$.*

*Proof.* There certainly exists such $a \in A$ that $a \circ D = D' \circ f$ holds on a small open set $U$ of $M$. Now $a \circ D$ and $D' \circ f$ are both $A$-maps of $M$ into $X$ which agree on an open set $U$, and hence are identical maps by proposition 1.1.

In particular, if $f = id$ on $M$, Proposition 1.2 says that developing maps defined on $M$ are essentially unique up to $A$.

Hence $A$-structures on a simply connected manifold $M$ can be para metrized by the equivalence classes of developing maps $\{D\}$, i.e., $D' \in \{D\}$ iff $D' = a \circ D$ for some $a \in A$. Note also that an $A$-structure on simply connected $M$ defines an immersion $D : M \to X$, a developing map, and conversely any immersion $D : M \to X$ defines an $A$-structure on $M$ by the pull-back structure.

Let $M$ be a connected $A$-manifold. We will fix a universal covering space $\tilde{M}$ of $M$ and the deck transformation group $\Pi$. The pull-back $A$-structure gives rise to a developing map $D : \tilde{M} \to X$, and any $A$-map $f$ of $\tilde{M}$ into itself will assign a unique $a \in A$ such that $a \circ D = D \circ f$ by Proposition 1.2. We will denote this unique $a$ by $\rho_D(f)$ so that $\rho_D(f) \circ D = D \circ f$. Let $A_D(\tilde{M})$ be the group of $A$-diffeomorphisms of $\tilde{M}$. Then $\rho_D : A_D(\tilde{M}) \to A$ is clearly a homomorphism since $\rho_D(f \circ g) \circ D = D \circ f \circ g = \rho_D(f) \circ D \circ g = \rho_D(f) \circ \rho_D(g) \circ D$ and an element of $A$ is uniquely determined by local data. The pull-back $A$-structure on $\tilde{M}$ is $\Pi$-periodic and hence $\Pi \subset A_D(\tilde{M})$. The restriction of $\rho_D$ on $\Pi$ is called the *holonomy representation* of an $A$-structure of $M$.

Note that if we use another developing map $D' = a \circ D$, $a \in A$, the holonomy representation with respect to $D'$ will be $\rho_D(\tau) = \rho_{a \circ D}(\tau) = a \circ \rho_D(\tau) \circ a^{-1} = c_a \circ \rho_D(\tau)$, $\tau \in \Pi$, where $c_a$ is the conjugation by $a$.

Indeed, $a \circ \rho_D(\tau) \circ D = a \circ D \circ \tau = D' \circ \tau = \rho_{D'}(\tau) \circ D' = \rho_{D'}(\tau) \circ a \circ D$ implies that $a \circ \rho_D(\tau) = \rho_{D'}(\tau) \circ a$ since they agree locally.

Let $M'$ be another $A$-manifold with a universal covering $\tilde{M}'$ and the deck transformation group $\Pi'$. Suppose we have a $A$-diffeomorphism $f : M \to M'$. Choose a lifting of $f$, $\tilde{f} : \tilde{M} \to \tilde{M}'$, then $\tilde{f}$ will induce a unique $a \in A$ such that $D' \circ \tilde{f} = a \circ D$ by Proposition 1.2. Now if we let $D'' = a \circ D = D' \circ \tilde{f}$, then $\rho_{D''} = c_a \circ \rho_D$ by the above note. Furthermore,

$$\begin{aligned} \rho_{D'}(\tilde{f} \circ \tau) \circ D' = D' \circ \tilde{f} \circ \tau = D'' \circ \tau = \rho_{D''}(\tau) \circ D'' \\ = \rho_{D''}(\tau) \circ D' \circ \tilde{f} = \rho_{D''}(\tau) \circ \rho_{D'}(\tilde{f}) \circ D' \end{aligned}$$

implies $\rho_{D''}(\tau) = \rho_{D'}(\tilde{f} \circ \tau \circ \tilde{f}^{-1}) = \rho_{D'} \circ c_{\tilde{f}}(\tau)$, where $c_{\tilde{f}}$ is the conjugation by $\tilde{f}$. Hence we have

$$c_a \circ \rho_D = \rho_{D''} = \rho_{D'} \circ c_{\tilde{f}}.$$

Note that $c_{\tilde{f}} : \Pi \to \Pi'$ is an isomorphism. Since $M$ and $M'$ are diffeomorphic, identifying $\Pi$ and $\Pi'$, this argument shows that if two $A$-structures on $M$ represented by developing maps $D$ and $D'$ are equivalent (by an $A$-diffeomorphism), then the associated holonomy representations $\rho_D$ and $\rho_{D'}$ define a same equivalent class $[\rho_D] = [\rho_{D'}] \in A \backslash \mathrm{Hom}(\Pi, A)/\mathrm{Aut}(\Pi)$, where $A$ acts on $\mathrm{Hom}(\Pi, A)$ on the left through conjugation and $\mathrm{Aut}(\Pi)$ acts on $\mathrm{Hom}(\Pi, A)$ as composition on the right. Note that these two actions commute trivially. Therefore we have a well-defined map $\Psi$ from the moduli space of $A$-structures on $M$ into the representation space $A \backslash \mathrm{Hom}(\Pi, A)/\mathrm{Aut}(\Pi)$ via holonomy representation. Of course, then the basic problems will be to determine the image and the fiber of $\Psi$. In general, these are quite difficult problems and we will discuss the fiber of $\Psi$ in the subsequent sections especially for $A$-structures whose developments are covering maps.

## 2. A parametrization of $A$-structures at $D$

From now on, we will consider only $A$-structure on $M$ whose development $D : \tilde{M} \to X$ is a covering map. Call such structure a *covering $A$-structure* in short. Since $\tilde{M}$ has pull-back $A$-structure, $D$

is "Π-periodic", i.e., $\Pi \subset A_D(\tilde{M})$ = the group of $A$-diffeomorphisms of $\tilde{M}$. Let $\Delta$ be the deck transformation group of the covering map $D : \tilde{M} \to X$. Then clearly we have the following short exact sequence of Lie groups :

$$1 \to \Delta \xrightarrow{i} A_D(\tilde{M}) \xrightarrow{\rho_D} A \to 1, \tag{2.1}$$

where $i$ is the inclusion map and $\rho_D$ is as defined in the previous section. In fact, (2.1) defines a unique Lie group structure on $A_D(\tilde{M})$ so that $\rho_D$ becomes a covering homomorphism with $\ker \rho_D = \Delta$.

We will fix a development $D$ and then will parametrize $A$-structures on $M$ by diffeomorphisms of $\tilde{M}$ as follows. Suppose we have a covering $A$-structure whose development is $D' : \tilde{M} \to X$. Then there exists a lifting $f : \tilde{M} \to \tilde{M}$ of identity $1_X : X \to X$ so that the following diagram commutes.

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{f} & \tilde{M} \\ {\scriptstyle D'}\downarrow & & \downarrow{\scriptstyle D} \\ X & \xrightarrow{1_X} & X \end{array}$$

Clearly $f$ is an $A$-map and $D' = D \circ f$ $(= f^*D)$. Since $D'$ is Π-periodic, for each $\tau \in \Pi$, $f \circ \tau \circ f^{-1} = c_f(\tau)$ is an $A$-diffeomorphism, and hence $c_f(\Pi) \subset A_D(\tilde{M})$. Note that if we use another lifting $f' = \delta \circ f$, $\delta \in \Delta$, then $c_{f'} = c_\delta \circ c_f : \Pi \to A_D(\tilde{M})$. Conversely, given any $f \in \mathrm{Diffeo}(\tilde{M})$ with $c_f(\Pi) \subset A_D(\tilde{M})$, $D' = D \circ f$ will define an $A$-structure on $\tilde{M}$ which is Π-periodic, i.e., each $\tau \in \Pi$ is an $A$-map, and hence defines an $A$-structure on $M$. Let's denote $M$ with $A$-structure $D' = D \circ f$ by $(M, f)$. Thus we can parametrize covering $A$-structures on $M$ by

$$\mathcal{F}_D = \{ f \in \mathrm{Diffeo}(\tilde{M}) \,|\, c_f(\Pi) \subset A_D(\tilde{M}) \}.$$

Let $N(\Pi) = N_{\mathrm{Diffeo}(\tilde{M})}(\Pi)$ denote the normalizer of Π in the diffeomorphism group $\mathrm{Diffeo}(\tilde{M})$ of $\tilde{M}$.

PROPOSITION 2.1. (a) *$(M, f) = (M, g)$, i.e., $id : (M, f) \to (M, g)$ is an $A$-diffeomorphism if and only if there is $\alpha \in A_D$ such that $g = \alpha \circ f$.*

(b) $(M,f)$ *and* $(M,g)$ *are A-diffeomorphic if and only if there exist* $\alpha \in A_D(\tilde{M})$ *and* $h \in N(\Pi)$ *such that* $g \circ h = \alpha \circ f$.

*Proof.* Let $\bar{h} : (M,f) \to (M,g)$ be an $A$-diffeomorphism. Then a lifting of $\bar{h}$, $h : \tilde{M} \to \tilde{M}$ is an $A$-diffeomorphism and normalizes $\Pi$. By proposition 1.2, there exists $a \in A$ such that $a \circ (D \circ f) = (D \circ g) \circ h$. Hence $D \circ (g \circ h \circ f^{-1}) = a \circ D$ and this shows that $\alpha = g \circ h \circ f^{-1} \in A_D(\tilde{M})$. The converse is obvious by reversing the argument. this proves (b), and (a) follows by letting $h$ = identity.

The above proposition suggests us to define an action of $A_D = A_D(\tilde{M})$ on $\mathcal{F}_D$ as left multiplication and an action of $N(\Pi)$ on $\mathcal{F}_D$ as right multiplication. Note that for $\alpha \in A_D$ and $f \in \mathcal{F}_D$, we have $\alpha \circ f \in \mathcal{F}_D$ since $c_{\alpha \circ f} = c_\alpha \circ c_f : \Pi \to A_D$, and that for $f \in \mathcal{F}_D$ and $h \in N(\Pi)$, $f \circ h \in \mathcal{F}_D$ since $c_{f \circ h} = c_f \circ c_h : \Pi \to A_D$. Note also that $N(\Pi) \subset \mathcal{F}_D$ and $A_D \subset \mathcal{F}_D$ since $\Pi \subset A_D$. Therefore $A_D \backslash \mathcal{F}_D$ can be viewed as the space of covering $A$-structures on $M$ and $A_D \backslash \mathcal{F}_D / N(\Pi)$ be *the moduli space of covering A-structures on M parametrized at D*, which will be denoted as $\mathcal{M}_D$.

Now let's see how holonomy representation looks like in this setting. Define $\psi : \mathcal{F}_D \to \mathrm{Hom}(\Pi, A)$ by $\psi(f) = \rho_D \circ c_f : \Pi \to A_D \to A$. Then $(\rho_D, \psi) : (A_D, \mathcal{F}_D) \to (A, \mathrm{Hom}(\Pi, A))$ is equivariant since $\psi(af) = \rho_D \circ c_{af} = \rho_D \circ c_a \circ c_f = c_{\rho_D(a)} \circ \rho_D \circ c_f = c_{\rho_D(a)} \circ \psi(f)$, and hence $\psi$ induces a quotient map : $A_D \backslash \mathcal{F}_D \to A \backslash \mathrm{Hom}(\Pi, A)$. Let $c : N(\Pi) \to \mathrm{Aut}(\Pi)$ be the conjugation defined by $c(f) = c_f$. Then $(\psi, c) : (\mathcal{F}_D, N(\Pi)) \to (\mathrm{Hom}(\Pi, A), \mathrm{Aut}(\Pi))$ is also equivariant. Indeed, $\psi(f \circ h) = \rho_D \circ c_{fh} = \rho_D \circ c_f \circ c_h = \psi(f) \circ c_h$. This shows that $\psi$ induces a map $\bar{\psi} : \mathcal{F}_D / N(\Pi) \to \mathrm{Hom}(\Pi, A)/\mathrm{Aut}(\Pi)$ and hence induces $\Psi : A_D \backslash \mathcal{F}_D / N(\Pi) \to A \backslash \mathrm{Hom}(\Pi, A)/\mathrm{Aut}(\Pi)$. Again the fore mentioned basic questions are to study the image and fiber of $\Psi$.

Let's pause at this point for a while to see what happens for "complete" case, i.e., when the developing map $D$ is a diffeomorphism. In this case, we may identify $\tilde{M}$ with $X$ via $D$ and all other covering developments become diffeomorphisms since $X$ must be simply connected. Hence $\mathcal{M}_D$ is the moduli space of complete $A$-structures on $M$. With this identification of $\tilde{M}$ with $X$, $A_D = A$ and $\rho_D$ becomes identity map. Then $\mathcal{F}_D = \{f \in \mathrm{Diffeo}(X) \mid c_f : \Pi \to A\}$ and $\psi : \mathcal{F}_D \to \mathrm{Hom}(\Pi, A)$ is defined simply by $\psi(f) = c_f$.

PROPOSITION 2.2("COMPLETE CASE"). *If $c : N(\Pi) \to Aut(\Pi)$ is onto, then $\bar{\psi} : \mathcal{F}_D/N(\Pi) \to Hom(\Pi, A)/Aut(\Pi)$ is 1-1.*

*Proof.* Suppose $\bar{\psi}\{f\} = \bar{\psi}\{g\}$. That is $c_f = \psi(f) = \psi(g) \circ \phi = c_g \circ \phi$ for some $\phi \in \mathrm{Aut}(\Pi)$. By the hypothesis, there is $h \in N(\Pi)$ such that $c_h = \phi$. Then $c_f = c_g \circ c_h$ implies $c_h = c_g^{-1} \circ c_f = c_{g^{-1} \circ f}$, and hence $g^{-1} \circ f \in N(\Pi)$.

COROLLARY 2.3("COMPLETE CASE"). *If $c : N(\Pi) \to Aut(\Pi)$ is onto, then $\Psi : \mathcal{M}_D = A \backslash \mathcal{F}_D/N(\Pi) \to A \backslash Hom(\Pi, A)/Aut(\Pi)$ is 1-1.*

*Proof.* This follows from the above proposition and a simple observation about the group action, which we will record as a Lemma below for later use also.

LEMMA 2.4. *Let $X$ be a $G$-space and $Y$ be an $H$-space. If $\phi : G \to H$ is a surjective homomorphism and $f : X \to Y$ is an injective equivariant map (via $\phi$), then the canonical induced map $\bar{f} : G\backslash X \to H\backslash Y$ is injective.*

*Proof.* Let's denote the $G$-orbit of $x \in X$ by $\bar{x}$ and similarly for $H$-orbit of $y \in Y$ by $\bar{y}$. $\overline{f(x)} = \bar{f}(\bar{x}) = \bar{f}(\bar{y}) = \overline{f(y)}$ implies that $f(x) = h \cdot f(y)$, $h \in H$. Since $\phi$ is surjective, $h = \phi(g)$, $g \in G$, and $f(x) = h \cdot f(y) = \phi(g) \cdot f(y) = f(g \cdot y)$ implies $x = g \cdot y$ by injectivity of $f$, whence $\bar{x} = \bar{y}$.

Let's go back to the more general case when $D$ is a covering map. In this case, we obtain corresponding faithful maps using $\mathrm{Hom}(\Pi, A_D)$ instead of $\mathrm{Hom}(\Pi, A)$. Let $c : \mathcal{F}_D \to \mathrm{Hom}(\Pi, A_D)$ be the conjugation map $c(f) = c_f$ and $Z(\Pi)$ be the centralizer of $\Pi$ in $\mathrm{Diffeo}(\tilde{M})$.

PROPOSITION 2.5. *$c : \mathcal{F}_D/Z(\Pi) \to Hom(\Pi, A_D)$ is injective.*

*Proof.* For $z \in Z(\Pi)$, $c_z = id$ and $c_{fz} = c_f \circ c_z = c_f$. This shows that the conjugation map $c$ induces a well-defined map on $\mathcal{F}_D/Z(\Pi)$. If $c_f = c_g$, $c_{f^{-1}g} = c_f^{-1} \circ c_g = id$ and so $f^{-1} \circ g \in Z(\Pi)$.

Furthermore, $c : (A_D, \mathcal{F}_D/Z(\Pi)) \to (A_D, \mathrm{Hom}(\Pi, A_D))$ is equivariant and hence induces a 1-1 map (by Lemma 2.4),

$$\bar{c} : \mathcal{T}_D = A_D \backslash \mathcal{F}_D/Z(\Pi) \to A_D \backslash \mathrm{Hom}(\Pi, A_D).$$

The space $\mathcal{T}_D$ will be called *the deformation space (of covering A-structures) parametrized at D*. Note that if we let $Q = N(\Pi)/Z(\Pi)$, then $\mathcal{T}_D/Q = \mathcal{M}_D = A_D \backslash \mathcal{F}_D / N(\Pi)$. Also notice that we have an exact sequence,

$$1 \to Z(\Pi) \to N(\Pi) \xrightarrow{c} \text{Aut}(\Pi).$$

Again by Lemma 2.4, we have

PROPOSITION 2.6. *If* $c : N(\Pi) \to Aut(\Pi)$ *is onto, then* $(\bar{c}, c) : (\mathcal{T}_D, Q) \to (A_D \backslash Hom(\Pi, A_D), Aut(\Pi))$ *is equivariant and induces an 1-1 map* : $\mathcal{M}_C \to A_D \backslash Hom(\Pi, A_D)/Aut(\Pi)$.

This proposition shows that in order to determine the fiber of $\Psi : \mathcal{M}_D \to A \backslash \text{Hom}(\Pi, A)/\text{Aut}(\Pi)$, we want to know the relation between $\text{Hom}(\Pi, A)$ and $\text{Hom}(\Pi, A_D)$.

## 3. Hom($\Pi, A_D$) and Hom($\Pi, A$)

Recall (2.1) that we have a short exact sequence of Lie groups,

$$1 \to \Delta \xrightarrow{i} A_D \xrightarrow{\rho} A \to 1 \tag{3.1}$$

where $\rho = \rho_D$ and $\Delta = \ker \rho$ is also the deck transformation group of a covering development $D : \tilde{M} \to X$. From this, it is not hard to imagine "exactness" of associated Hom-sequence of sets,

$$1 \to \text{Hom}(\Pi, \Delta) \xrightarrow{i_*} \text{Hom}(\Pi, A_D) \xrightarrow{\rho_*} \text{Hom}(\Pi, A). \tag{3.2}$$

Indeed, $i_*$ is clearly 1-1 and $\rho_* \circ i_*$ maps to a trivial representation. For any $\phi \in \text{Hom}(\Pi, A_D)$ with $\rho_*(\phi) = \rho \circ \phi$ being trivial, certainly $\phi \in \text{Hom}(\Pi, \Delta)$. We want to be more precise about the fibration nature of (3.2) to show that $\rho_*$ is a covering map with fiber $\text{Hom}(\Pi, \Delta)$ under some conditions. For a Lie group $G$, we will always give compact-open topology for $\text{Hom}(\Pi, G)$ and discrete topology for $\Pi$ and $\Delta$. First of all, it is easy to see that the continuity of $\rho_*$ from the following well-known observation.

LEMMA 3.1. *Let $f : Y \to Z$ be a continuous map of topological spaces and $X$ be a locally compact, Hausdorff space. Then $f_* : Y^X \to Z^X$ given by $f_*(\phi) = f \circ \phi$, $\phi \in Y^X$ is continuous with respect to compact-open topology on $Y^X$ and $Z^X$.*

*Proof.* Since $X$ is locally compact, Hausdorff, $f_*$ is continuous if the associated map $\bar{f}_* : X \times Y^X \to Z$ given by $\bar{f}_*(x, \phi) := f_*(\phi)(x)$ is continuous. Note that $\bar{f}_* = f \circ ev : X \times Y^X \to Y \to Z$, where $ev(x, \phi) = \phi(x)$. As is well-known, $ev$ is continuous and follows the continuity of $\bar{f}_*$.

Let's assume for simplicity that $A_D$ is connected deferring the general case to a subsequent paper. Since $\Delta$ is a discrete normal subgroup of a connected Lie group $A_D$, $\Delta$ is central and (3.1) is a central extension. Now $\mathrm{Hom}(\Pi, \Delta)$ becomes an abelian group and define an action of $\mathrm{Hom}(\Pi, \Delta)$ on $\mathrm{Hom}(\Pi, A_D)$ by $(d \cdot \Phi)(\tau) = d(\tau) \cdot \phi(\tau)$ for $d \in \mathrm{Hom}(\Pi, \Delta)$ and $\phi \in \mathrm{Hom}(\Pi, A_D)$. Note that $d \cdot \phi$ is a homomorphism : $\Pi \to A_D$ since $\Delta$ is central and $\Pi \subset A_D$. Furthermore, this action is clearly free. The fiber of $\rho_*$ is exactly $\mathrm{Hom}(\Pi, \Delta)$-orbit and the quotient map to the orbit space can be identified with $\rho_*$. Indeed, if $\rho \circ \phi = \rho \circ \phi'$ for $\phi, \phi' \in \mathrm{Hom}(\Pi, A_D)$, then $d(\tau) = \phi'(\tau) \cdot \phi(\tau)^{-1} \in \Delta$ is a homomorphism : $\Pi \to A_D$ since $\Delta$ is central. Now the following theorem looks obvious.

THEOREM 3.2. *Suppose $A_D$ is connected and $\Pi$ is finitely generated. Then the action : $Hom(\Pi, \Delta) \times Hom(\Pi, A_D) \to Hom(\Pi, A_D)$ given by $\mu(d, \phi) = d \cdot \phi$ is a covering action, and the covering projection can be identified with the map $\rho_* : Hom(\Pi, A_D) \to \rho_*(Hom(\Pi, A)) \subset Hom(\Pi, A)$ as a set function.*

*Proof.* From the above discussion, it suffices to show that for each $\phi \in \mathrm{Hom}(\Pi, A_D)$, we can choose an open neighborhood $W$ of $\phi$ such that $d \cdot W \cap W$ is empty for all non-trivial $d \in \mathrm{Hom}(\Pi, \Delta)$. Since $\Delta$ is a discrete subgroup of $A_D$, there exists an open neighborhood $V_1$ of $e$ in $A_D$ disjoint from the non-trivial element of $\Delta$ and $\rho|_{V_1}$ is a diffeomorphism onto an open set $\rho(V_1) \subset A$. Let $V \subset V_1$ be a neighborhood of $e$ with the property $V \cdot V^{-1} \subset V_1$. Let $U(\tau) = V \cdot \phi(\tau)$ and $W = \cap_{i=1}^n S(\tau_i, U(\tau_i))$, where $\{\tau_1, \cdots, \tau_n\}$ is a set of generators of $\Pi$, and $S(K, U)$ stands for subbasic open set in the compact-open topology for $A_D^\Pi$ whose element sends a compact $K \subset \Pi$ into an open

$U \subset A_D$. If $\psi \in d \cdot W \cap W$, $\psi = d \cdot \psi'$ for some $\psi' \in W$, and this implies that $d(\tau_i)\psi'(\tau_i) = \psi(\tau_i) \in U(\tau_i) = V \cdot \phi(\tau_i)$ and $\psi'(\tau_i) \in V \cdot \phi(\tau_i)$ for $i = 1, 2, \cdots, n$. Now $d(\tau_i) = \psi(\tau_i)\psi'(\tau_i)^{-1} = v \cdot \phi(\tau_i) \cdot (v' \cdot \phi(\tau_i))^{-1} = v \cdot v'^{-1}$ for some $v, v' \in V$. Hence $d(\tau_i) \in V_1 \cap \Delta = \{e\}$ and $d(\tau_i) = e$ for $i = 1, \cdots, n$, i.e., $d$ is trivial.

REMARK 3.3. In general, it is not clear whether the quotient map $q$ of $\mathrm{Hom}(\Pi, A_D)$ onto its orbit space $\mathrm{Hom}(\Pi, A_D)/\mathrm{Hom}(\Pi, \Delta)$ can be topologically identified with $\rho_*$. In the following commutative diagram, theorem 3.2 says $q$ is a covering map and the induced map $\bar{\rho}_*$ is a continuous injection.

$$\begin{array}{ccc} \mathrm{Hom}(\Pi, A_D) & \xrightarrow{\rho_*} & \mathrm{Hom}(\Pi, A) \\ {\scriptstyle q}\downarrow & \nearrow{\scriptstyle \bar{\rho}_*} & \\ \mathrm{Hom}(\Pi, A_D)/\mathrm{Hom}(\Pi, \Delta) & & \end{array}$$

$\bar{\rho}_*$ becomes an embedding for instance if $A_D$ is compact, since the variety of representation $\mathrm{Hom}(\Pi, A_D)$ is compact. If $A_D$ is abelian, or $\Pi$ is a free or free abelian, it can be modified in the proof of Theorem 3.2 to show that $\rho_*(W)$ is open in $\rho_*(\mathrm{Hom}(\Pi, A_D))$ and hence $\bar{\rho}_*$ becomes an embedding and we can identify $q$ with $\rho_*$ topologically.

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