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ON MUTATIONS OF ASSOCIATIVE ALGEBRAS

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1. Introduction

Let A be an arbitrary associative algebra over a field F, multiplication denoted by juxtaposition, and let p, q be two fixed elements of A. Then a new algebra is derived from A by using the same vector space structure of A but defining a new multiplication

$$(*) \qquad x * y = xpy - yqx$$

for x, y in A. The resulting algebra is denoted by $A(p,q)$ and called the (p,q)-*mutation* of the algebra A.

These algebras have been studied by several mathematicians and physicists (see [6] and references there in).

The overall objective of this paper is the investigation of the structure of the mutations of (left or right) artinian associative algebras. We will find out the existence for these algebras of a largest solvable ideal. The quotient algebra modulo this ideal will be shown to be a direct sum of ideals which are either simple algebras of a determined form (not necessarily mutation algebras) or semisimple Lie algebras with simple derived algebras.

The second section will be devoted to find necessary and sufficient conditions for a mutation algebra to be prime or simple. Then in the third section the mutations of simple artinian associative algebras will be studied. Finally, those previous results will be used in the fourth section to investigate mutations of arbitrary artinian associative algebras.

In what follows, the characteristic of the fields will be always assumed to be different from 2.

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We will make use of several other algebras derived from old ones. First of all, if we take $q = 0$ in $(*)$, the resulting algebra is associative and is termed the *p-homotope* of A, denoted by $A(p)$.

Next, let B be any nonassociative algebra with multiplication denoted by xy. Associated with B there are an anticommutative algebra B^-, defined on the same vector space but multiplication given by

$$[x, y] = xy - yx,$$

and a commutative algebra B^+ with multiplication

$$x \circ y = \frac{1}{2}(xy + yx).$$

The algebra B is termed *Lie-admissible* in case B^- is a Lie algebra, and is termed *Jordan-admissible* in case B^+ is a Jordan algebra. It is well-known that associative algebras are Lie and Jordan-admissible.

A straightforward calculation shows that $A(p,q)^- = A(p+q)^-$ and $A(p,q)^+ = A(p-q)^+$. Hence any mutation algebra is also Lie and Jordan-admissible.

2. Prime and simple mutation algebras

We begin with an easy statement: if $A(p,q)$ is prime or simple, so is A. Actually, any ideal of A is also an ideal of $A(p,q)$, hence the simplicity of $A(p,q)$ implies the simplicity of A. Now, if there are nonzero ideals B_1, B_2 of A with $B_1B_2 = 0$, then either $B_2B_1 = 0$ and, in consequence, $B_1 * B_2 = 0$ in $A(p,q)$, or $0 \neq B = B_2B_1 \subseteq B_1 \cap B_2$ verifies $B^2 = B * B = 0$. Thus, if $A(p,q)$ is prime, so is A.

It is shown in [7] that for invertible elements p and $q, p \neq q$, in the associative algebra A, the mutation $A(p,q)$ is prime or simple if and only if so is A. On the other hand, if $A(p,q)$ has a unit element then, as proved in [1], $A(p,q)$ is prime or simple if and only if so is A.

Since the existence of unit element in $A(p,q)$ is equivalent to $p - q$ being invertible and $(p-q)^{-1}p$ central, the results above provide sufficient conditions on the elements p and q of a prime or simple associative algebra that guarantee the mutation algebra $A(p,q)$ to be prime or simple.

We will find in this section necessary and sufficient conditions on the elements p and q of a prime (respectively simple) associative algebra

A for the mutation $A(p,q)$ to be prime (respectively simple), provided $p \neq q$.

In case $p = q$, the algebra $A(p,q)$ coincides with $A(p)^-$, so it is the minus algebra of an associative algebra. If, for instance, A is a finite dimensional algebra, then it can be shown that $A(p)^-$ is never prime nor simple (see Theorem 3.6 and the remarks following it).

So the case $p = q$ will be excluded in most of our considerations in this section.

The next Lemma is essentially the same as in [7; Lemma 3.1]:

LEMMA 2.1. *Let A be a prime associative algebra and let p, q be any fixed elements of A with $p + q \neq 0$. Then:*

1) *The subspace C spanned by the set $\{xpy + yqx : x, y \in A\}$ contains a nonzero ideal of A.*
2) *If B is a nonzero ideal of $A(p,q)$, then there is a nonzero ideal D of A such that $D(p-q)B$ and $B(p-q)D$ are contained in B.*
3) *If B is a nonzero ideal of $A(p,q)$, then either it contains a nonzero ideal of A or $(p-q)B(p-q) = 0$.*

For an element u of the associative algebra A, we shall denote by $r(u)$ (respectively $l(u)$) the right (respectively left) annihilator of u in A. That is, $r(u) = \{x \in A : ux = 0\}$ and $l(u) = \{x \in A : xu = 0\}$.

LEMMA 2.2. *Let A be a prime associative algebra, let p, q be any fixed elements of A with $p + q \neq 0 \neq p - q$, and let B be an ideal of $A(p,q)$ with $(p-q)B(p-q) = 0$. Then $B(p-q) \subseteq r(p) \cap r(q)$ and $(p-q)B \subseteq l(p) \cap l(q)$.*

Proof. By Lemma 2.1 there is a nonzero ideal D of A such that $B(p-q)D \subseteq B$. Therefore $(p-q)(ApB(p-q)D)(p-q) \subseteq (p-q)(A * B)(p-q) + (p-q)B(p-q)DqA(p-q) = 0$, so $pB(p-q) = 0$ by primeness. Hence $B(p-q) \subseteq r(p)$. The other assertions are proved in a similar way.

PROPOSITION 2.3. *Let A be a prime associative algebra, let p, q be any fixed elements of A with $p \neq q$ and let B be a nonzero ideal of $A(p,q)$. Then either B contains a nonzero ideal of A or B is contained in the set $R(p,q) = \{x \in A : pxp = pxq = qxp = qxq = 0\}$.*

Proof. In case $p \neq -q$ we may assume by the previous lemmas that $pB(p-q) = qB(p-q) = (p-q)Bp = (p-q)Bq = 0$. Now for $u \in B$ and $x \in A$

$$0 = p(u * x)(p-q) = pupx(p-q) - pxqu(p-q) = pupx(p-q).$$

Hence $ApBpA(p-q)A = 0$ and, by primeness, $pBp = 0$. The other assertions follow in the same way.

In case $p = -q(\neq 0)$, $A(p,q) = A(p)^+$. Following the argument in [7; page 911], any ideal of $A(p)^+$ contains $ApupupA$ for u in B. Now if $pupup = 0$ for all $u \in B$ we get $pupApup = 0$ for all $u \in B$ and, by the primeness of A, $pup = 0$ for all $u = B$.

Taking into account that the homotope $A(p)$ equals the mutation algebra $A(p,0)$, Proposition 2.3 and [7; page 913] we get:

LEMMA 2.4. *Let A be an associative algebra and p a fixed element of A. Then:*

1) *$A(p)$ is prime if and only if A is prime and p is not a zero divisor of A.*
2) *A^+ is prime if and only if so is A.*

The next Theorem represents the main result of this section:

THEOREM 2.5. *Let A be an associative algebra and let p, q be fixed elements of A with $p \neq q$. Then $A(p,q)$ is prime (respectively simple) if and only if A is prime (respectively simple) and $R(p,q) = 0$.*

Proof. Let us assume that A is prime, $R(p,q) = 0$ and $A(p,q)$ is not prime. Then there are nonzero ideals B_1, B_2 of $A(p,q)$ with $B_1 * B_2 = 0$. If $B_1 \cap B_2 = 0$ we would obtain, by Proposition 2.3, nonzero ideals D_1, D_2 of A with $D_1 \cap D_2 = 0$, so $D_1 D_2 = 0$, a contradiction. If $B = B_1 \cap B_2 \neq 0$, again by Proposition 2.3 there is a nonzero ideal D of A with $D \subseteq B$, so $D * D = 0$. Then for any u, v in D we get $upv = vqu$.

Now, for u, v in D and x in $r(p-q)$ we have

$$(xu)(p-q)v = (xu)pv - (xu)qv = vq(xu) - vp(xu) = -v(p-q)xu = 0,$$

so $r(p-q)D(p-q)D = 0$ and the primeness of A imply $r(p-q) = 0$. In the same way we have $l(p-q) = 0$, so $p-q$ is not a zero divisor of A.

Hence, by Lemma 2.4, $A(p-q)^+$ is prime. But $A(p-q)^+ = A(p,q)^+$ and D is a nonzero ideal of $A(p,q)^+$ which squares to zero, a contradiction.
The converse and the simple case are easy.

Notice that if there are elements α, β in the center $Z(A)$ of A such that $\alpha p + \beta q$ is not a zero divisor, then

$$R(p,q) \subseteq \{x \in A : (\alpha p + \beta q)x(\alpha p + \beta q) = 0\} = 0.$$

Hence we get the next Corollary with covers [7; Theorem 3.1] and [1; Corollary 1.5 and Theorem 1.9]:

COROLLARY 2.6. *Let A be an associative algebra and let p, q be any fixed elements in A with $p \neq q$. Assume that there is an element in $Z(A)p + Z(A)q$ which is not a zero divisor. Then $A(p,q)$ is prime or simple if and only if so is A.*

If A is an associative algebra and Γ its centroid (see [5; Chapter X]), then it is clear that Γ is contained in the centroid $\Gamma_{p,q}$ of $A(p,q)$ for any p, q of A. Now, if A is simple we have that $A \otimes_\Gamma K$ is simple for any field extension K of F. Assume that $R(p,q) = 0$ and $p \neq q$. Since this condition does not depend on the ground field, Theorem 2.5 tells us that $A(p,q) \otimes_\Gamma K = (A \otimes_\Gamma K)(p \otimes 1, q \otimes 1)$ is simple for any field extension K of Γ, so Γ is the centroid of $A(p,q)$ too. Hence for $p \neq q$ we get that $A(p,q)$ is central simple if and only if A is central simple and $R(p,q) = 0$.
For prime algebras we also get the coincidence of the centroids:

THEOREM 2.7. *Let A be a prime associative algebra and let p, q be elements in A with $p \neq q$ and $R(p,q) = 0$. Then the centroids of A and $A(p,q)$ coincide.*

Proof. Let Γ be the centroid of A and $\Gamma_{p,q}$ the centroid of $A(p,q)$, so $\Gamma \subseteq \Gamma_{p,q}$. Let $\gamma \in \Gamma_{p,q}$, we will write x^γ for the image of the element x under γ. Let us consider the subspace H spanned by the set $\{(xy)^\gamma - xy^\gamma : x, y \in A\}$. For x, y, z in A, $z*((xy)^\gamma - xy^\gamma) = z*(xy)^\gamma - z*(xy^\gamma) = (z*(xy))^\gamma - z*(xy^\gamma) = (zpxy)^\gamma - (xyqz)^\gamma - zpxy^\gamma + xy^\gamma qz$.
But $z * y^\gamma = (z*y)^\gamma$, so $zpy^\gamma - y^\gamma qz = (zpy)^\gamma - (yqz)^\gamma$. Hence $z * ((xy)^\gamma - xy^\gamma) = (zpxy)^\gamma - (xyqz)^\gamma - zpxy^\gamma + xzpy^\gamma - x(zpy)^\gamma + x(yqz)^\gamma = [((zpx)y)^\gamma - (zpx)y^\gamma] - [(x(yqz))^\gamma - x(yqz)^\gamma] + [(x(zpy))^\gamma -$

$x(zpy)^\gamma] - [((xzp)y)^\gamma - (xzp)y^\gamma]$ which is in H. Therefore $A * H \subseteq H$ and, in the same way, we obtain $H * A \subseteq H$. Thus H is an ideal of $A(p,q)$.

Let us assume that $H \neq 0$. By Proposition 2.3 there is a nonzero ideal D of A such that $D \subseteq H$. Let us consider the set $S = \{x \in A : (x(p-q)z)^\gamma = x(p-q)z^\gamma$ and $(z(p-q)x)^\gamma = z^\gamma(p-q)x\ \forall z \in A\}$.

Notice that for x, y and z in A, $(x*z)*y - x*(z*y) + (yqx)*z + z*(xpy) = z(p-q)(xpy+yqx)$, and since $((x*z)*y - x*(z*y)+(yqx)*z + z*(xpy))^\gamma = (x*z^\gamma)*y - x*(z^\gamma*y) + (yqx)*z^\gamma + z^\gamma*(xpy)$, we get $(z(p-q)(xpy+yqx))^\gamma = z^\gamma(p-q)(xpy+yqx)$. Symmetrically $((xpy+yqx)(p-q)z)^\gamma = (xpy+yqx)(p-q)z^\gamma$. Therefore $xpy+yqx \in S$ for all x, y in A.

If $p+q \neq 0$, Lemma 2.1,1) gives us an ideal I of A contained in S. If, on the contrary, $p+q=0$, then $S = \{x \in A : (xpz)^\gamma = xpz^\gamma$ and $(zpx)^\gamma = z^\gamma px\ \forall z \in A\}$, so S is easily seen to be a subalgebra of the homotope algebra $A(p) = A(p,0)$, and since $xpy - ypx \in S$ for all x, y in A, we get that S is also an ideal of $A(p)^-$ which contains $[A(p), A(p)]$. Since $A(p) = A(p,0)$ is prime (Theorem 2.5), [4; Theorem 3] shows that either S is contained in the center of $A(p)$ or S contains a nonzero ideal of $A(p)$. In the first case [4; Lemma 1] shows that $A(p)$ is commutative, so $A(p,q) = A(2p)^+ = A(2p) = A(2p,0)$ and we are back in the case $p+q \neq 0$. If S contains a nonzero ideal of $A(p) = A(p,0)$, then, by Proposition 2.3, S contains a nonzero ideal of A.

The conclusion of the last paragraph is that there is always a nonzero ideal I of A such that for any $x \in I$ and $z \in A$, $(x(p-q)z)^\gamma = x(p-q)z^\gamma$ and $(z(p-q)x)^\gamma = z^\gamma(p-q)x$. Now, for x, y in A and u in I we have $(xu(p-q)y)^\gamma = ((xu)(p-q)y)^\gamma = xu(p-q)y^\gamma = x(u(p-q)y)^\gamma$, so $(xy)^\gamma = xy^\gamma$ for any $x \in A$ and $y \in I(p-q)A$. Since A is prime, $A(p-q)I(p-q)A \neq 0$. Let us take a nonzero element $u \in (p-q)I(p-q)A$. Then for $x, y \in A$:

$$
\begin{aligned}
(xyu)^\gamma &= (xy)^\gamma u \text{ since } u \in (p-q)I \text{ and } I \subseteq S,\\
(xyu)^\gamma &= x(yu)^\gamma \text{ since } yu \in I(p-q)A,\\
(yu)^\gamma &= y^\gamma u \text{ since } u \in (p-q)I \text{ and } I \subseteq S
\end{aligned}
$$

Hence $((xy)^\gamma - xy^\gamma)u = (xyu)^\gamma - x(yu)^\gamma = (xyu)^\gamma - (xyu)^\gamma = 0$. In consequence $Hu = 0$, so $Du = 0 = D(AuA)$ and, since A is prime, we arrive at $u = 0$, a contradiction.

Therefore $H = 0$ and $(xy)^\gamma = xy^\gamma$ for any x, y in A. Similarly we prove that $(xy)^\gamma = x^\gamma y$ for any x, y in A, so γ is in the centroid of A.

3. Mutations of simple artinian associative algebras

We will show in this section that a mutation $A(p,q)$ of a simple artinian associative algebra A, with $p \neq q$, contains a maximal nilpotent ideal R and a simple subalgebra S such that $A(p,q) = R \oplus S$. The subalgebra S has a quite definite form. In case $p = q$, $A(p,q)$ is a Lie algebra which contains a maximal solvable ideal. The quotient modulo this ideal verifies that its derived algebra is simple.

We begin with a characterization of the ideal $R(p,q) = \{x \in A : pxp = pxq = qxp = qxq = 0\}$, which will turn out to be the largest nilpotent ideal of $A(p,q)$ in case $p \neq q$.

LEMMA 3.1. *Let A be a semisimple artinian associative algebra and let p, q be any elements of A. Then $R(p,q) = (r(p) \cap r(q)) + (l(p) \cap l(q))$.*

Proof. Since any right or left ideal of A is generated by an idempotent [2], we get $Ap + Aq = Ae$ and $pA + qA = fA$, with $e^2 = e$ and $f^2 = f$. Now, for $x \in R(p,q)$ we have $exf = 0$ so $xf \in r(e) = r(p) \cap r(q)$, $x - xf \in l(f) = l(p) \cap l(q)$ and $x = xf + (x - xf) \in (r(p) \cap r(q)) + (l(p) \cap l(q))$. That $(r(p) \cap r(q)) + (l(p) \cap l(q)) \subseteq R(p,q)$ is obvious.

THEOREM 3.2. *Let A be a simple artinian associative algebra and let p, q be any fixed elements of A with $p \neq q$. Then $A(p,q) * A(p,q) = A(p,q)$, $R(p,q)$ is the only maximal ideal of $A(p,q)$, $R(p,q)$ is nilpotent and there is a simple subalgebra S of $A(p,q)$ such that $A(p,q) = R(p,q) \oplus S$.*

Proof. As in Lemma 2.1,1), changing q by $-q$, we get that the subspace spanned by $\{xpy - yqx : x, y \in A\}$ contains a proper ideal of A, so for A simple this subspace is the whole A. This proves that $A(p,q) * A(p,q) = A(p,q)$. It is straightforward to see that $R(p,q) * (R(p,q) * R(p,q)) = 0 = (R(p,q) * R(p,q)) * R(p,q)$, so $R(p,q)$ is a nilpotent ideal of $A(p,q)$. In particular $R(p,q) \neq A(p,q)$. Proposition 2.3 and the simplicity of A prove that $R(p,q)$ is the only maximal ideal of $A(p,q)$.

Finally, let $Ap+Aq = Ae$ and $pA+qA = fA$, with $e^2 = e$ and $f^2 = f$. Then $R(p,q) = (r(p)\cap r(q)) + (l(p)\cap l(q)) = r(e) + l(f) = (1-e)A + A(1-f)$. Therefore $A = eA+(1-e)A = e(Af+A(1-f))+(1-e)A \subseteq eAf+(1-e)A+A(1-f) = eAf+R(p,q)$. Moreover, if $x \in eAf\cap R(p,q)$, then $x \in R(p,q) = r(e) + l(f)$, so $x = x_1 + x_2$, with $ex_1 = 0 = x_2f$. But $x \in eAf$ so $x = exf = ex_1f + ex_2f = 0$. Thus $eAf \cap R(p,q) = 0$ and $A(p,q) = R(p,q)\oplus eAf$. Since $(eAf)A(eAf) \subseteq eAf$, eAf is closed under the product in $A(p,q)$. Hence $S = eAf$ is a subalgebra of $A(p,q)$ which is simple since so is $A(p,q)/R(p,q)$.

Let us determine more closely what the simple subalgebras that appear in the last Theorem are like. Since A is a simple artinian associative algebra, there is a division algebra D and an integer n such that A is isomorphic to the algebra of matrices $M_n(D)$ ([2]). We shall identify then A with the algebra $\mathrm{End}_D(V)$ of endomorphisms of a vector space V of dimension n over D (we will consider the action of D on V on the right).

Let p,q be elements of A with $p \neq q$. If $r(p)\cap r(q) = l(p)\cap l(q) = 0$, then $A(p,q)$ is simple and we are done. Otherwise, let $\{v_1,\cdots,v_n\}$ be a basis of V over D such that $\{v_{r+1},\cdots,v_n\}$ is a basis of $\ker p\cap\ker q$. Then p and q are represented by matrices

$$p = \begin{pmatrix} p_{11} & \dots & p_{1r} & 0 & \dots & 0 \\ p_{21} & \dots & p_{2r} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & 0 & \ddots & \vdots \\ p_{n1} & \dots & p_{nr} & 0 & \dots & 0 \end{pmatrix}$$

$$q = \begin{pmatrix} q_{11} & \dots & q_{1r} & 0 & \dots & 0 \\ q_{21} & \dots & q_{2r} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ q_{n1} & \dots & q_{nr} & 0 & \dots & 0 \end{pmatrix}$$

and $r(p)\cap r(q) = \{x \in End_D(V) : x(V) \subseteq \ker p\cap\ker q\}$, so $r(p)\cap r(q)$

consists of the endomorphisms represented by matrices of the form

$$\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ a_{r+1,1} & \cdots & a_{r+1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix}$$

Now, the set $l(p) \cap l(q)$ consists of those matrices x such that each row $(\alpha_1, \cdots, \alpha_n)$ verifies:

$$(\alpha_1, \cdots, \alpha_n) \begin{pmatrix} p_{11} & \cdots & p_{1r} & q_{11} & \cdots & q_{1r} \\ p_{21} & \cdots & p_{2r} & q_{21} & \cdots & q_{2r} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ p_{n1} & \cdots & p_{nr} & q_{n1} & \cdots & q_{nr} \end{pmatrix} = 0$$

We can solve this system and get a set of indices $L = \{i_1, \cdots, i_s\}$, such that any solution is given by

$$\alpha_{i_1} = \sum_{j \notin L} \alpha_j d_1^j$$

$$\vdots$$

$$\alpha_{i_s} = \sum_{j \notin L} \alpha_j d_s^j$$

Let B be the set of $n \times n$ matrices over D with 0's in the last $n-r$ rows and in the columns with indices not in L. Then it is easy to see that $A(p,q) = R(p,q) \oplus B$ and B is a subalgebra of $A(p,q)$, which is simple by the last Theorem.

But for $x, y \in B$, when performing the product $x*y = (xp)y-(yq)x$, only the rows with indices in L of p and q are relevant. Therefore, if we denote by $\hat{p}$ the $s \times r$ matrix over D formed by the intersection of the first r columns and the rows with indices in L of p, and the same for q, we get that the subalgebra B is isomorphic to the algebra $M_{r\times s}(D)$, of $r \times s$ matrices over D, with multiplication given by

$$x * y = x\hat{p}y - y\hat{q}x$$

where $\hat{p}, \hat{q} \in M_{s\times r}(D)$.

Moreover $\hat{p}$ and $\hat{q}$ are such that $\hat{p} \neq \hat{q}$, there is no matrix in $M_{r\times s}(D)$ with $\hat{p}x = \hat{q}x = 0$ and there is no matrix in $M_{r\times s}(D)$ with $x\hat{p} = x\hat{q} = 0$. The last two conditions are equivalent to $\{x \in D^r : \hat{p}x^t = \hat{q}x^t = 0\} = 0 = \{x \in D^s : x\hat{p} = x\hat{q} = 0\}$, where x^t denotes the transpose of x. All this can be seen directly from the construction of $\hat{p}$ and $\hat{q}$ or by appealing to B being a simple algebra. The algebra $M_{r\times s}(D)$ with the product $x * y = x\hat{p}y - y\hat{q}x$ will be denoted $D_{r,s}(\hat{p}, \hat{q})$.

Conversely, let us assume that $u, v \in M_{s\times r}(D)$, $u \neq v$ and $\{x \in D^r : ux^t = vx^t = 0\} = 0 = \{x \in D^s : xu = xv = 0\}$. If $n \geq \max\{r, s\}$ and we take $A = M_n(D)$,

$$p = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix} \quad q = \begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix},$$

then it can be checked that $r(p) \cap r(q)$ consists of those matrices in $M_n(D)$ with the first r rows equal to zero, and $l(p) \cap l(q)$ consists of those matrices in $M_n(D)$ with the first s columns equal to zero. Moreover, $A(p, q)/R(p, q)$ is isomorphic to $D_{r,s}(u, v)$.

Summarizing we get:

THEOREM 3.3. *Let A be a simple artinian associative algebra, $A \cong M_n(D)$ for some division algebra D, and let p, q be elements of A with $p \neq q$. Then there is a simple subalgebra S of $A(p, q)$ such that $A(p, q) = R(p, q) \oplus S$ and S is isomorphic to $D_{r,s}(u, v)$ for some $r, s \leq n$ and $u, v \in M_{s\times r}(D)$ such that $u \neq v$ and $\{x \in D^r : ux^t = vx^t = 0\} = 0 = \{x \in D^s : xu = xv = 0\}$. Conversely, any such algebra $D_{r,s}(u, v)$ is simple and isomorphic to $A(p, q)/R(p, q)$ for some simple artinian associative algebra and elements $p, q \in A$ with $p \neq q$.*

In case p and q are linearly dependent we get $r = s$ so $A(p, q)/R(p, q)$ is again a mutation algebra. But for p and q linearly independent it may well happen that $r \neq s$ as the following example shows:

EXAMPLE 3.4. Let us take $A = M_2(F), p = \begin{pmatrix} p_1 & 0 \\ p_2 & 0 \end{pmatrix}$, $q = \begin{pmatrix} q_1 & 0 \\ q_2 & 0 \end{pmatrix}$, with (p_1, p_2) and (q_1, q_2) linearly independent in F^2. Then $r(p) \cap r(q) = \{\begin{pmatrix} 0 & 0 \\ \alpha & \beta \end{pmatrix} : \alpha, \beta \in F\}$ and $l(p) \cap l(q) = 0$. So $R(p, q) = r(p) \cap r(q)$ and

$$A(p, q)/R(p, q) \cong F_{1,2}\left(\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}\right).$$

This is a simple algebra of dimension 2. If it were isomorphic to a mutation algebra $C(a,b)$, then C would be a simple associative algebra of dimension 2, so C would be a quadratic field extension of F and $C(a,b)$ would be commutative. However, in $F_{1,2}\left(\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}\right)$, $(1,0)*(0,1) = -(q_2,p_1) \neq (p_2,-q_1) = (0,1)*(1,0)$.

It remains to study the case in which $p = q$. Of course, if $p = q = 0$, then $A(p,q) * A(p,q) = 0$ and there is nothing to do. In other case $A(p,p) = A(p)^-$ and $R(p,p) = \{x \in A : pxp = 0\}$, which will be denoted by $R(p)$. From Theorem 3.3 we know that there is a subalgebra S of $A(p) = A(p,0)$ such that $A(p) = R(p) \oplus S$. Moreover, the subalgebra S with the product in $A(p)$ is isomorphic to $D_{r,s}(\hat{p},0)$ with $\{x \in D^r : \hat{p}x^t = 0\} = 0 = \{x \in D^s : x\hat{p} = 0\}$. This implies that $r = s$ and that $\hat{p}$ is invertible in $M_r(D)$. Hence $D_{r,s}(\hat{p},0) = M_r(D)(\hat{p})$, which is isomorphic to $M_r(D)$. So we get:

THEOREM 3.5. *Let A be a simple artinian associative algebra, $A \cong M_n(D)$ for some division algebra D, and let p be a nonzero element of A. Then there is a simple subalgebra S of $A(p)$ with $A(p,p) = R(p) \oplus S$, $R(p)$ is a nilpotent ideal of $A(p,p)$ and S, with the product in $A(p,p) = A(p)^-$, is isomorphic to $M_r(D)^-$ for some $r \leq n$.*

From [3; Chapter 1] we know that any ideal of $M_r(D)^-$ either is contained in the center $Z(M_r(D)) = Z(D)I$ (I is the identity matrix), or contains the derived subalgebra $[M_r(D), M_r(D)]$. Moreover, if $M_r(D)$ is not commutative, that is, if D is not a field or $r > 1$, then

$$[M_r(D), M_r(D)]/(Z(M_r(D)) \cap [M_r(D), M_r(D)]) \cong \\ ([M_r(D), M_r(D)] + Z(M_r(D)))/Z(M_r(D))$$

is a simple Lie algebra. Hence:

THEOREM 3.6. *Let A be a simple artinian associative algebra and let p be a nonzero element of A. Then the ideal $J(p)$ of $A(p,p)$, given by $J(p)/R(p) = Z(A(p,p)/R(p))$ (or, equivalently, $J(p) = R(p) \oplus Z(S)$) is the largest solvable ideal of $A(p,p)$. If $J(p) \neq A(p,p)$, then the quotient $B(p) = A(p,p)/J(p)$ is a semisimple Lie algebra with $[B(p), B(p)]$ simple.*

EXAMPLE 3.7. Let $A = M_2(F)$ and $p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $R(p) = \{x \in A : pxp = 0\} = \{\begin{pmatrix} 0 & \alpha \\ \beta & \gamma \end{pmatrix} : \alpha, \beta, \gamma \in F\}$. Hence $A(p,p)/R(p)$ is a Lie algebra of dimension 1, so $J(p) = A(p,p)$. Therefore, $A(p,p)$ is solvable, but it is not nilpotent since $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. This shows that $J(p)$ needs not be nilpotent.

In general there is no subalgebra T of $A(p,p)$ with $A(p,p) = J(p) \oplus T$. Think, for example, of $A = M_n(F)$ for n a prime number and F a field of characteristic n. Then if p is the identity matrix, then $A(p,p) = A^-$, $J(p) = FI \subseteq [A, A]$ and $J(p)$ is not complemented by any subalgebra of A^-. However, if D is a finite dimensional division algebra over a field of characteristic 0, then it is well known that $M_r(D) = Z(M_r(D)) \oplus [M_r(D), M_r(D)]$, and $[M_r(D), M_r(D)]$ is a simple Lie algebra of type A (See [5]). Then:

COROLLARY 3.7. *Let A be a finite dimensional simple associative algebra over a field of characteristic 0 and p a nonzero element of A. Then if $J(p) \neq A(p,p)$, there is a simple subalgebra T of $A(p,p)$ with $A(p,p) = J(p) \oplus T$.*

4. The solvable radical of a mutation algebra

Let A be an artinian associative algebra and let p, q be elements of A. Then the Jacobson radical $R(A)$ of A is a nilpotent ideal of A, so it is a nilpotent ideal of $A(p,q)$ too. Moreover $A/R(A) = A_1 \oplus \cdots \oplus A_n$, with the A_i's simple algebras.

Then $p + R(A) = p_1 + \cdots + p_n$ and $q + R(A) = q_1 + \cdots + q_n$, with p_i, q_i in A_i, $i = 1, \ldots, n$, and $A(p,q)/R(A) = A_1(p_1, q_1) \oplus \cdots \oplus A_n(p_n, q_n)$.

We have seen in the last section that each $A_i(p_i, q_i)$ has a largest solvable ideal $J(p_i, q_i)$, which is $R(p_i, q_i)$ if $p_i \neq q_i$ (Theorem 3.2), or $J(p_i)$ if $p_i = q_i$ (Theorem 3.6). Then the ideal $J(p,q)$ given by $J(p,q)/R(A) = J(p_1, q_1) \oplus \cdots \oplus J(p_n, q_n)$ is the largest solvable ideal of $A(p,q)$ and will be called the *solvable radical* of $A(p,q)$. With the results of the last section we have:

THEOREM 4.1. *Let A be an artinian associative algebra and let p, q be elements of A. Then $A(p,q)$ has a largest solvable ideal $J(p,q)$ and* *the quotient $A(p,q)/J(p,q)$ is a direct sum of ideals, each one being isomorphic either to a simple algebra $D_{r,s}(u,v)$, for positive integers r, s, division algebra D and $u \neq v$ (as in Theorem 3.3), or to a semisimple Lie algebra of the form $M_r(D)^-/Z(M_r(D))$, for some positive integer r and division algebra D.*

COROLLARY 4.2. *Let A be an artinian associative algebra and let p, q be any fixed elements of A. Then $J(p,q) = 0$ if and only if A is semisimple, $l(p) \cap l(q) = 0 = r(p) \cap r(q)$ and there is no maximal ideal M of A with $p - q \in M$.*

Notice that for artinian associative or alternative algebras and for finite dimensional Lie or Malcev algebras over fields of characteristic 0, the solvable radical equals the intersection of the maximal ideals of A which do not contain A^2 (see the proof of Theorem 4.5). Hence

COROLLARY 4.3. *Let A be an artinian associative algebra and let p, q be any fixed elements of A. Then $J(p,q) = R(A)$ if and only if there is no maximal ideal M of A not containing A^2 such that $p-q \in M$ and the sets $\{x \in A : px, qx \in R(A)\}$ and $\{x \in A : xp, xq \in R(A)\}$ are contained in $R(A)$.*

COROLLARY 4.4. *Let A be an artinian associative algebra and let p, q be any fixed elements of A. If $A(p,q)$ has a unit element then $J(p,q) = R(A)$.*

Proof. If $A(p,q)$ has a unit element, then $p-q$ is invertible (see [1]), so Corollary 4.3 applies.

Finally we shall give some sufficient conditions for the radical $J(p,q)$ to be nilpotent. Example 3.7 shows that this is not always the case.

THEOREM 4.5. *Let A be an artinian associative algebra and let p, q be any fixed elements of A. If $p - q \notin M$ for any maximal ideal M of A not containing A^2, then $J(p,q)$ is nilpotent.*

Proof. We know that $R(A) \subseteq J(p,q)$. On the other hand $R(A)$ is contained in any maximal ideal M of A not containing A^2 for, in other case, $A = R(A) + M$ and $A^2 \subseteq R(A)^2 + M$. But $A^2 \not\subseteq M$, so $M \subset R(A)^2 + M$ and $A = R(A)^2 + M$. This implies, in the same way, that $A = R(A)^4 + M$ and, eventually, we shall arrive to $A = M$, since $R(A)$ is nilpotent.

Now $A/R(A) = A_1 \oplus \cdots \oplus A_n$, with the A_i's simple ideals, $p+R(A) = p_1 + \cdots + p_n$, $q+R(A) = q_1 + \cdots + q_n$, with $p_i, q_i \in A_i$, $i = 1, \ldots, n$. Our conditions imply that $p_i \neq q_i$ for all i, so $J(p,q)/R(A) = R(p_1, q_1) \oplus \cdots \oplus R(p_n, q_n)$ from the remarks preceding Theorem 4.1. Hence for any $x \in J(p,q)$ we have that pxp, pxq, qxp and qxq are all in $R(A)$ (actually $J(p,q) = \{x \in A : pxp, pxq, qxp, qxq \in R(A)\}$). Any element in $J(p,q) * J(p,q) * \overset{(2m+1)}{\cdots} * J(p,q)$ (with any order of parenthesis) is a sum of elements of the form $x = x_1 u_1 x_2 u_2 x_3 u_3 \cdots u_{2m} x_{2m+1}$, with $x_i \in J(p,q)$, $i = 1, \ldots, 2m+1, u_i = p$ or q, $i = 1, \ldots, 2m$. But $x = x_1(u_1 x_2 u_2) x_3 \cdots x_{2m-1}(u_{2m-1} x_{2m} u_{2m}) x_{2m+1}$, and each $u_i x_{i+1} u_{i+1}$ belongs to $R(A)$. Hence $x \in R(A)^m$, so if we take m large enough so that $R(A)^m = 0$, we get that any product of $2m+1$ elements of $J(p,q)$ is 0, so $J(p,q)$ is a nilpotent ideal of $A(p,q)$.

We know that if we drop the condition of $p - q$ not belonging to M for any maximal ideal of A not containing A^2, then $J(p,q)$ may not be nilpotent (Example 3.7).

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