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# FINITELY BASED MODULAR LATTICE VARIETIES

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## 1. Introduction

In K. Baker [1] it is shown that the variety $M$ of all modular lattices has $2^{\aleph_0}$ subvarieties. It follows that there exists a variety of modular lattices that it is not finitely based. Here our investigation is concerned with the question wether the join of two finitely based lattice varieties is finitely based. The answer to question is not always affirmative. It was first shown by K. Baker that the join of two finitely based lattice varieties need not be finitely based. Here the original question for modular lattice varieties is investigated under certain conditions. Actually we obtain the following results.

THEOREM 1.1. *Let $V$ and $V'$ be finitely based lattice varieties. If $B_k \notin V$ and $A_2 \notin V'$, then $V + V'$ is finitely based.*

THEOREM 1.2. *Let $V$ and $V'$ be finitely based lattice varieties. If $A_1 \notin V$ and $\bar{E} \notin V'$, then $V + V'$ is finitely based.*

The rest of this paper is divided into two sections. In Section 2 we will give some preliminary definitions and facts. And finally in last section we shall prove the theorem 1.1, 1.2 and state their corollaries. For standard concepts and facts from lattice theory we refer the reader to Grätzer [4]. However we use $+$ and $\times$ instead of $\vee$ and $\wedge$ for the lattice operations.

## 2. Preliminaries

A class $K$ of algebras is *finitely based* if it is the class of all models of some finite set of identities.

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DEFINITION 2.1. A class $K$ of first order $L$-structure is *elementary* if there is a set $\sum$ of first order formulas such that

$$A \in K \text{ if and only if } \sum \text{ holds in } A.$$

And an elementary class $K$ is *strictly elementary* if $\sum$ can be taken to be finite.

Now we will review a well-known theorem in universal algebra, which is very useful in this paper.

THEOREM 2.2. *Let $V$ and $V'$ be varieties. Then the following are equivalent.*

(1) *$V + V'$ is finitely based.*
(2) *$V + V'$ is strictly elementary.*
(3) *The complement of $V + V'$ is closed under ultraproduct.*

*Here the complements can be taken relative to any finitely based supervariety $U$.*

In Jónsson [6] and [8], the following criterion for membership in the join of two congruence distributive varieties is obtained.

THEOREM 2.3. *Suppose $U$ is a congruence distributive variety, and let $V$ and $V'$ be subvarieties of $U$ defined, relative to $U$, by the identities $\alpha = \beta$ and $\gamma = \delta$, respectively. Then for an algebra $A$ the following are equivalent.*

(1) $A \in V + V'$
(2) *$A \leq B \times B'$ with $B \in V$ and $B' \in V'$.*
(3) *$\theta \cap \theta' = O_A$, the null congruence relation in $A$, where $\theta, \theta' \in$ Con$(A)$, the lattice of all congruences over $A$, are the smallest congruence relations with $A/\theta \in V$ and $A/\theta' \in V'$, respectively.*
(4) *$con(\alpha(\mu), \beta(\mu)) \cap con(\gamma(\nu),\ \theta(\nu)) = 0$ for all $\mu,\ \nu \in {}^{\omega}A$.*

This theorem applies in particular to lattice varieties. The closer study of congruence relations on lattices is based Dilworth's concepts of projectivity. Consider two quotients $a/b$ and $c/d$ in a lattice $L$. If $a + b = c$ and $ad = b$, then we say that *$a/b$ trasposes up onto $c/d$* and that *$c/d$ trasposes down onto $a/b$* (in symbols, $a/b \nearrow c/d$ and $c/d \searrow a/b$).

If there exists a sequence of quotients $a/b = a_0/b_0, a_1/b_1, \cdots, a_n/b_n = c/d$ such that for $i = 0, 1, 2, \cdots, n-1$, $a_i/b_i \nearrow a_{i+1}/b_{i+1}$ or $a_i/b_i \searrow a_{i+1}/b_{i+1}$, then we say that $a/b$ *projects onto* $c/d$ in $n$ steps. By *projective distance* between two quotients $a/b$ and $c/d$ - in symbols $P(a/b, c/d)$ - we mean that the smallest nonnegative integer $n$ such that some nontrivial subquotients $a'/b'$ of $a/b$ and $c'/d'$ of $c/d$ are projective to each other in $n$ steps. If no such $n$ exists, then we write $P(a/b, c/d) = \infty$. We write $Pd(a/b, c/d) = n$ if $P(a/b, c/d) = n$ and $a/b$ is projective to $c/d$. For modular lattices, principal congruences can be described in terms of projectivities. Therefore the criterion for membership in $V + V'$ in Theorem 2.3 can be expressed in terms of this notion for modular lattice varieties.

COROLLARY 2.4. *Suppose $U$ is a modular lattice variety. Let $V$ and $V'$ be subvarieties of $U$ defined, relative to $U$, by the identities $\alpha = \beta$ and $\gamma = \delta$ respectively, where the inclusions $\beta \leq \alpha$ and $\delta \leq \gamma$ hold in $U$. Then a lattice $L \in U$ belongs to $V + V'$ if and only if two nontrivial subquotients of $\alpha(\mu)/\beta(\mu)$ and $\gamma(\nu)/\delta(\nu)$ with $\mu, \nu \in {}^{\omega}L$ never project onto each other.*

By a *strongly normal* sequence of transposes in $L$ we mean a (finite) sequence of quotients

$$a_0/b_0, a_1/b_1, \cdots, a_n/b_n \tag{1}$$

such that if for $0 \leq k \leq n$, $a_{k-1}/b_{k-1} \nearrow a_k/b_k \searrow a_{k+1}/b_{k+1}$, $a_k = a_{k-1} + a_{k+1}$ and $a_{k-1}a_{k+1} < b_k$ or if $a_{k-1}/b_{k-1} \searrow a_k/b_k \nearrow a_{k+1}/b_{k+1}$, $b_k = b_{k-1}b_{k+1}$ and $b_{k-1} + b_{k+1} > a_k$. Suppose (1) is a strongly normal sequence of transposes in $L$. We see from Figure 1 that if a quotient $a_{k-1}/b_{k-1}$ is nontrivial, the figure contains a nontrivial diamond

$$D_k = [v_k < x_k, y_k, z_k < u_k]$$

where

$$D_k = [b_{k-1} + b_{k+1} < a_{k-1} + b_{k+1}, b_k, b_{k-1} + a_{k+1} < a_k]$$

if

$$a_{k-1}/b_{k-1} \nearrow a_k/b_k \searrow a_{k+1}/b_{k+1}$$

and

$$D_k = [b_k < b_{k-1}a_{k+1}, a_k, a_{k-1}b_{k+1} < a_{k-1}a_{k+1}]$$

if

$$a_{k-1}/b_{k-1} \searrow a_k/b_k \nearrow a_{k+1}/b_{k+1}.$$

By this argument, a strongly normal sequence of transposes (1) in $L$ generates a sequence of $(n-1)$ diamonds $D_1, D_2, \cdots, D_{n-1}$. This said to be the associated sequence of diamonds.

In Jónsson [6], it was proved that if two quotients in a modular lattice $L$ project onto each other in $n$ steps, then there exist nontrivial subquotients of them which project strongly normally onto each other in $\leq n$ steps. Therefore, with each sequence of projectivities there is always an associated sequence of diamonds. We must now investigate how these diamonds fit together. First, we define some notations. Given two diamonds

$$D_i = [v_i < x_i, y_i, z_i < u_i], \quad i = 1, 2.$$

We say that $D_1$ *transposes down onto* $D_2$ (in symbols $D_1 \searrow_{(1)} D_2$) or that $D_2$ *transposes up onto* $D_1$ (in symbols $D_2 \nearrow_{(1)} D_1$) if $u_1/v_1 \searrow u_2/v_2$, and under this transposition the vertices $x_1, y_1, z_1$ are mapped onto the corresponding vertices $x_2, y_2, z_2$ (see Figure 3). Also, we say that $D_1$ *translates up onto* $D_2$ (in symbols $D_1 \nearrow_{(2)} D_2$) and that $D_1$ *translates down onto* $D_2$ (in symbols $D_1 \searrow_{(2)} D_2$) if $u_1/z_1 \nearrow x_2/v_2$ and if $z_1/v_1 \searrow u_2/v_2$, respectively (see Figure 4). Note that $D_1 \nearrow_{(2)} D_2$ does not imply $D_2 \searrow_{(2)} D_1$. If $D = [v < x, y, z < u]$ is a diamond, then $D^*$ is defined to be the diamond $[v < z, x, y < u]$. So $D_1 \searrow_{(1)} D_2^*$ means that $u_1/v_1 \searrow u_2/v_2$, $x_1u_2 = z_2$, $y_1u_2 = x_2$ and $z_1u_2 = y_2$. The investigation of how these associated diamonds fit together was done by D. X. Hong [5]. That contains the following useful theorem. We call it Hong's Theorem in this paper.

THEOREM 1.6 (HONG'S THEOREM). *Let $a/b$ and $c/d$ be nontrivial quotients in a modular lattice such that $P(a/b, c/d) = n$, $2 \leq n < \infty$. Then some nontrivial subquotients $a'/b'$ and $c'/d'$ of $a/b$ and $c/d$, respectively, can be connected by a strongly normal sequence of transposes*

$$a'/b' = a_0/b_0, a_1/b_1, \cdots, a_n/b_n = c'/d'$$

*such that the associated diamonds* $D_1, D_2, \cdots, D_{n-1}$ *satisfy*

(i) $D_k \nearrow_{(1)} D^*_{k+1}$ *or* $D_k \nearrow_{(2)} D_{k+1}$ *if* $a_k/b_k \nearrow a_{k+1}/b_{k+1}$ *and* $D_k \searrow_{(1)} D^*_{k+1}$ *or* $D_k \searrow_{(2)} D_{k+1}$ *if* $a_k/b_k \searrow a_{k+1}/b_{k+1}$, $k = 1, 2, \cdots, n-2$.

(ii) *If* $D_k \nearrow_{(1)} D^*_{k+1}$ *or* $D_k \searrow_{(1)} D^*_{k+1}$, *then* $D_k = D^*_{k+1}$, $k = 2, \cdots, n-2$.

(iii) *If* $D_k \nearrow_{(1)} D^*_{k+1}$ *or* $D_k \searrow_{(1)} D^*_{k+1}$, *then it can not happen that* $D_{k+1} \searrow_{(1)} D^*_{k+2}$ *or* $D_{k+1} \nearrow_{(1)} D^*_{k+2}$, *respectively.*

*If the conditions (i), (ii) and (iii) are satisfied, then we refer to the strongly normal sequence of transposes in Hong's Theorem as a Hong sequence.*

*Finally, we introduce some lattices and notation used later. Let* $A_1$, $A_2$, *and* $A_3$ *be the lattices pictured in Figure 5.*

## 3. Proofs of main theorems

By a *critical* quotient of a lattice $L$ we mean a quotient that is collapsed by every nontrivial congruence relation on $L$.

LEMMA 3.1 (JÓNSSON [8]). *Let* $V$ *and* $V'$ *be subvarieties of* $M$ *defined, relative to* $M$, *by identities* $\alpha = \beta$ *and* $\gamma = \delta$, *respectively, where the inclusions* $\beta \leq \alpha$ *and* $\delta \leq \gamma$ *hold in* $M$. *In order for* $V + V'$ *to be finitely based relative to* $M$, *it is necessary and sufficient that there exists a positive integer* $n$ *with the following property:*

$P(n)$: *For any* $L \in M$, *if there exist* $\mu, \nu \in {}^{\omega}L$ *such that a nontrivial subquotient of* $\alpha(\mu)/\beta(\mu)$ *projects onto a nontrivial subquotient or* $\gamma(\nu)/\delta(\nu)$, *then there exist* $\mu', \nu' \in {}^{\omega}L$ *such that a nontrivial subquotient of* $\alpha(\mu')/\beta(\mu')$ *projects onto a nontrivial subquotient of* $\gamma(\nu')/\delta(\nu')$ *in* $n$ *steps.*

NOTATION. For any modular lattice $L$, if there exists a nonnegative integer $n$ such that, for all $a, b, c, d \in L$ with $b < a$ and $d < c$, whenever $a/b$ projects onto $c/d$, then a nontrivial subquotient of $a/b$ projects onto a nontrivial subquotient of $c/d$ in $n$ steps, then the smallest such $n$ is denoted by $R(L)$. If no such $n$ exists, then we write $R(L) = \infty$. For a class $K$ of lattices, $R(K)$ denotes the supremum of $R(L)$ for $L \in K$. Also, let $\bar{x}$ denote the image of each $x \in L$ in the homomorphic image $\bar{L}$ of $L$, and we shall use this notion for any homomorphic images of a given lattice.

LEMMA 3.2. *Let $\bar{L}$ be a homomorphic image of a lattice $L$, and let $p/q$ be a prime quotient in $L$ with $\bar{q} < \bar{p}$. For all $a, b \in L$ with $b < a$ and for any nonnegative integer $n$, if $\bar{a}/\bar{b}$ projects onto $\bar{p}/\bar{q}$ in $n$ steps, then a nontrivial subquotient of $a/b$ projects onto $p/q$ in $(n+1)$ steps when $n > 0$, and in two steps $n = 0$.*

LEMMA 3.3. *Suppose the quotients $a/b$ and $c/d$ in a modular lattice are connected by a Hong sequence of length $n$. If the associated sequence of diamonds $D_i$, $i < n$, does not contain a subsequence of the form $D_{k-1} \nearrow_{(2)} D_k \searrow_{(2)} D_{k+1}$, $1 < k < n-1$, then it contains at most one subsequence of the form $D_{k-1} \searrow_{(2)} D_k \nearrow_{(2)} D_{k+1}$.*

*Proof.* The inclusion hold trivially if $n \leq 4$, so we assume that $n > 4$. Consider four cases.

Case(i) : $D_1 \nearrow_{(2)} D_2$: by our assumption $D_2 \searrow_{(1)} D_3^*$, and hence by Hong's Theorem, $D_3 \nearrow_{(2)} D_4$. By the same argument, $D_4 \searrow_{(1)} D_5^*$ and $D_5 \nearrow_{(2)} D_6$. Continuing in the same manner, we see that the condition $D_k \searrow_{(2)} D_{k+1}$, $k < n-1$, never holds.

Case(ii) : $D_1 \searrow_{(2)} D_2$: by Hong's Theorem, there are two subcases.

Subcase ($\mathrm{ii}_a$) : $D_2 \nearrow_{(1)} D_3^*$ : again using Hong's Theorem, we infer that $D_3 \searrow_{(2)} D_4$. This reduces the problem to the corresponding problem for a shorter sequence.

Subcase ($\mathrm{ii}_b$) : $D_2 \nearrow_{(2)} D_3$: by Case(i), there is no index $k > 1$ with $D_k \searrow_{(2)} D_{k+1}$.

Case(iii) : $D_1 \nearrow_{(1)} D_2^*$: by Hong's Theorem, we have $D_2 \searrow_{(2)} D_3$. The conclusion follows from Case(ii).

Case(iv) : $D_1 \searrow_{(1)} D_2^*$: by Hong's Theorem, we have $D_2 \nearrow_{(2)} D_3$. The conclusion follows from Case(i). The proof is complete.

Observation. If $A$ and $B$ are sublattices of a lattice $L$, and if a filter $F$ of $A$ projects up onto an ideal $I$ of $B$, then $A \cup B$ is a sublattice of $L$ containing $A$ as an ideal and $B$ as a filter.

This trivial but extremely useful observation is the basic for the classical Dilworth-Hall construction.

LEMMA 3.4. *Given a sequence of Hong's associated diamonds $D_1, D_2, \cdots, D_{2k-4}$ for $k > 2$ in a modular lattice $L$. If $D_1 \nearrow_{(2)} D_2$ (or $D_1 \searrow_{(2)} D_2$), and if the numbers below the arrows alternate, then the sublattice $L_0$ of $L$ generated by $D_1, D_2, \cdots, D_{2k-4}$ is finite and has the finite simple lattice $B_k$ pictured in Figure 6 as a homomorphic image.*

*Proof.* By our assumption, $D_{2t} = D_{2t+1}{}^*$, $0 < 2t < 2k-3$. Thus $D_1, D_2, D_4, \cdots, D_{2k-4}$ generates $L_0$. Trivially, $u_1/z_1$ is a filter of the sublattice $D_1$ of $L$ and $x_2/v_2$ is an ideal of the sublattice $D_2$ of $L$. Also, by assumption, $u_1/z_1 \nearrow x_2/v_2$. Hence by the Observation, $D_1 \cup D_2$ is a sublattice of $L$. Also it is trivial that $u_2/x_2$ is a filter of $D_1 \cup D_2$ and that $x_4/v_4$ is an ideal of the sublattice $D_4$ of $L$. By hypothesis, $D_3 \nearrow_{(2)} D_4$. Hence $u_3/z_3 \nearrow x_4/v_4$. Since $D_2 = D_3^*$, we have $u_2/x_2 = u_3/z_3$. Thus we have $u_2/x_2 \nearrow x_4/v_4$. Again, by the Observation, $D_1 \cup D_2 \cup D_4$ is a sublattice of $L$. Continuing in this manner, we have that $D_1 \cup D_2 \cup D_4 \cup \cdots \cup D_{2k-4}$ forms a sublattice $L_0$ of $L$ which is a homomorphic image of the lattice $B_k$ pictured in Figure 6. Therefore, $L_0$ is finite and has the finite simple lattice $B_k$ as a homomorphic image. The proof is complete.

LEMMA 3.5. *Let $B_k$ be the simple lattice of length $k$ pictured in Figure 6. Then $R(B_k) = 2k-1$.*

*Proof.* We use mathematical induction on $k$. Since $R(M_3) = 3$, the lemma is true for $k = 2$. Assume that it is true for $k = t$. Since the other case can be treated similarly, we may assume that the relation between the $t$-th diamond and $(t+1)-st$ diamond is as indicated in Figure 7, namely that $u_t/x_t$, $u_{t+1}/y_{t+1}$. Also, each prime subquotient of $u_t/v_t$ projects onto $u_{t+1}/y_{t+1}$ in $(2t+1)$ steps. Therefore the lemma is true for $k = t+1$. The proof is complete.

*Proof of Theorem 1.1.* Let $V$ and $V'$ be defined by the identities $\alpha = \beta$ and $\gamma = \delta$, respectively, relative to the variety $M$ of all modular lattices. We may assume that the inclusion $\beta \leq \alpha$ and $\delta \leq \gamma$ hold in every modular lattice.

In order to show that $V + V'$ is finitely based it is sufficient, by Theorem 2.2, to show that the complement $K$ of $V + V'$ in $M$ is a strictly elementary class. Consider any lattice $L \in M$. Let $\theta$ and $\theta'$ be the smallest conguence relations on $L$ with $L/\theta \in V$ and $L/\theta' \in V'$. Then by Theorem 2.3, $L \in K$ if and only if $\theta \cap \theta' \neq O_L$, the null congruence relation on $L$. In other words, $L \in K$ if and only if some nontrivial quotient in $L$ is collapsed by both $\theta$ and $\theta'$. In Jónsson [8], ideas from Baker used to express this property as a disjunction of infinitely many elementary properties. Since we are concerned here with modular lattice varieties, it is convenient notion of $Pd(a/b, c/d)$.

The condition referred to can be expressed as follows.

$P_n(L)$ : There are nontrivial quotients $a/b$ and $c/d$ in $L$ such that

(i) $a/b \leq \alpha(\mu)/\beta(\mu)$ for some $\mu \in {}^{\omega}L$
(ii) $c/d \leq \gamma(\nu)/\delta(\nu)$ for some $\nu \in {}^{\omega}L$
(iii) $Pd(a/b, c/d) = n$.

By Corollary 2.4, for any $L \in M$, $L \in K$ if and only if $P_n(L)$ holds for some natural number $n$. In other words $K$ is defined, relative to $M$, by the disjunction of infinitely many fomulas $P_n(L)$. We claim that if $P_n(L)$ holds for some $n \geq 4k+3$, then $P_n(L)$ holds for some $m < 4k+3$. Suppose $P_n(L)$ holds in $L$ for some $n \geq 4k+3$. Then by the definition of $Pd(a/b, c/d) = n$, there exists a Hong's sequence $a'/b' = a_0/b_0, a_1/b_1, \cdots, a_n/b_n = c'/d'$ for some nontrivial subquotients $a'/b'$ and $c'/d'$ of $a/b$ and $c/d$, respectively. Also by Hong's Theorem, there exists the associated sequence of diamonds $D_1, D_2, \cdots, D_{n-1}$. Then we have the following two cases.

(1) there exists a subsequence $D_i, D_{i+1}, D_{i+2}$ with $1 \leq i \leq n-2$ such that $D_i \nearrow_{(2)} D_{i+1} \searrow_{(2)} D_{i+2}$.
(2) There exists no such subsequence.

Case(1) : Let $q$ be the smallest positive integer with $1 \leq q < n-2$ such that a subsequence $D_q, D_{q+1}, D_{q+2}$ satisfies $D_q \nearrow_{(2)} D_{q+1} \searrow_{(2)} D_{q+2}$. We have the following two subcases.

$$q < 4k-5 \tag{1.1}$$
$$4k-5 \leq q \tag{1.2}$$

Case(1.1) : By the construction of the diamonds, we have $v_{q+1} = v_{q+2} + v_q$ and $z_q x_{q+2} = (u_q z_{q+1})(x_{q+1} u_{q+2}) = u_q u_{q+2}$. Hence $D_q \cup D_{q+1} \cup D_{q+2} \cup \{u_q u_{q+2}, u_q v_{q+2}, v_{q+2} u_q, v_q v_{q+2}\}$ forms a sublattice $L_0$ of $L$ which is a homomorphic image of the lattice $C$ pictured in Figure 8. Therefore $L_0$ contains the lattice $A_2$ pictured in Figure 5 as a homomorphic image. Also, $a_q/b_q$ is a prime quotient in $L_0$. Since $A_2$ is a simple lattice, $\bar{a}_q/\bar{b}_q$ is a critical quotient in $A_2$. Since $A_2 \notin V'$, $\gamma(\overline{\nu'}) > \delta(\overline{\nu'})$ for some $\nu' \in {}^{\omega}L_0$. Observe that $\gamma(\overline{\nu'}) = \overline{\gamma(\nu')}$ and $\delta(\overline{\nu'}) = \overline{\delta(\nu')}$. Since $R(A_2) \leq 6$, $\gamma(\overline{\nu'})/\delta(\overline{\nu'})$ projects onto $\bar{a}_q/\bar{b}_q$ in 6 steps. Since $a_q/b_q$ is a prime quotient in $L_0$, by Lemma 3.2, a prime subquotient of $\gamma(\nu')/\delta(\nu')$ projects onto $a_q/b_q$ in 7 steps. Thus $a/b$

projects onto a prime subquotient of $\gamma(\nu')/\delta(\nu')$ in $(q+7)$ steps. Since $q < 4k-5$, $q+7 < 4k+2$. Thus $P_m(L)$ holds for some $m < 4k+3$.

Case(1.2) : By Lemma 3.3, there is at most one subsequence $D_t$, $D_{t+1}, D_{t+2}$ with $0 < t \leq q-1$ such that $D_t \searrow_{(2)} D_{t+1} \nearrow_{(2)} D_{t+2}$. We have the following two subcases.

(1.2.1) there exists such a subsequence for $q-2k+4 < t \leq q-1$.

(1.2.2) there exists no such subsequences.

Take $p = t-2k+5$ if (1.2.1) holds, but $p = q-2k+5$ if (1.2.2) holds. By the assumptions of Case(1), Case(1.2), Case(1.2.1) and Case(1.2.2), for any $i$, with $p < i < p+2k-4$, neither $D_i \nearrow_{(2)} D_{i+1} \searrow_{(2)} D_{i+2}$ nor $D_i \searrow_{(2)} D_{i+1} \nearrow_{(2)} D_{i+2}$. Therefore, the subsequence $D_p, D_{p+1}, \cdots,$ $D_{p+2k-5}$ satisfies the condition that the numbers below arrows alternate and $D_p \nearrow_{(2)} D_{p+1}$ or $D_p \searrow_{(2)} D_{p+1}$. Let $L_1$ be the sublattice of $L$ generated by $D_p, D_{p+1}, \cdots, D_{p+2-5}$. Then by Lemma 3.4, we have a simple quotient lattice $\bar{L}_1$ of $L_1$ which is isomorphic to the lattice $B_k$ pictured in Figure 6. Hence by Lemma 3.5, $R(\bar{L}_1) = 2k-2$. Also, $a_{p+2k-5}/b_{p+2k-5}$ is a prime quotient in $L_1$. Since $\bar{L}_1$ is a simple lattice, $\bar{a}_{p+2k-5}/\bar{b}_{p+2k-5}$ is a critical quotient in $\bar{L}_1$. Since $\bar{L}_1 \cong B_k \notin V$, $\alpha(\overline{\mu'}) > \beta(\overline{\mu'})$ for some $\mu' \in {}^{\omega}\bar{L}_1$. Since $R(\bar{L}_1) = 2k-2$, $\alpha(\overline{\mu'})/\beta(\overline{\mu'})$ projects onto $\bar{a}_{p+2k-5}/\bar{b}_{p+2k-5}$ in $(2k-2)$ steps. Since $a_{p+2k-5}/b_{p+2k-5}$ is a prime quotient in $L_1$, by Lemma 3.2, a prime subquotient of $\alpha(\mu')/\beta(\mu')$ projects onto $a_{p+2k-5}/b_{p+2k-5}$ in $(2k-1)$ steps. Also by the argument of Case (1.1), a prime subquotient of $\gamma(\nu')/\delta(\nu')$ for some $\nu' \in {}^{\omega}L_0$ projects onto $a_q/b_q$ in 7 steps. Since $p = t-2k+5$ or $q-2k+5$, a prime subquotient of $\alpha(\mu')/\beta(\mu')$ projects onto a prime subquotient of $\gamma(\nu')/\delta(\nu')$ in $(4k+2)$ steps. Thus $P_m(L)$ holds in some $m < 4k+3$.

Case(2) : Take $p = n-2k+3$. By the argument of Case (1.2.2), $P_m(L)$ holds in some $m < 2k+3$.

By the above argument, we have already seen that for any $L \in M$, if $P_n(L)$ holds for some $n \geq 4k+3$, then $P_m(L)$ holds for $m < 4k+3$. Thus $K$ is defined by the disjunction of the formulas $P_n(L)$ with $n \leq 4k+2$, relative to $M$. Therefore $V+V'$ is finitely based. The proof is complete.

COROLLARY 3.6. *Let $V$ and $V'$ be finitely based modular lattice varieties. If $B_k \notin V$ and $A_3 \notin V'$, then $V+V'$ is finitely based.*

LEMMA 3.7. *A modular lattice $L$ generated by four diamonds $D_1$,*

*$D_2, D_3, D_4$ with the property that $D_1 \nearrow_{(2)} D_2 \searrow_{(2)} D_3 \nearrow_{(2)} D_4$ is a homomorphic image of the lattice $E$ pictured in Figure 9. Furthermore, $L$ has the finite simple lattice $\bar{E}$ pictured in Figure 10 as a homomorphic image.*

*Proof.* It is enough to show that $v_1 + v_4 = z_2 + x_4$ and $u_1u_4 = z_1x_3$. Examining the sublattice of $L$ generated by $D_1, D_2$ and $D_3$, we have that $v_1 + z_3 = z_2$ and $v_1u_3 + z_3 = u_3$. (See Figure 8). By $D_3 \nearrow_{(2)} D_4$, we have $u_3 + v_4 = x_4$. Therefore $z_2 + x_4 + v_1 + z_3 + u_3 + v_4 = v_1 + z_3 + v_1u_3 + z_3 + v_4$. Since $z_3 \leq v_4$ and $v_1u_3 \leq v_1$, we have $z_2 + x_4 = v_1 + v_4$. Dually, we have $u_1u_4 = z_1x_3$.

*The Proof of Theorem 1.2.* Let $V$ and $V'$ be defined be the identities $\alpha = \beta$ and $\gamma = \delta$, respectively, relative to the variety $M$. We may assume that the inclusion $\beta \leq a$ and $\delta \leq \gamma$ hold in every modular lattice.

In order to show that $V + V'$ is finitely based it is sufficient, by Theorem 2.2, to show that the complement $K$ of $V + V'$ in $M$ is a strictly elementary class. Consider any lattice $L \in M$. Let $\theta$ and $\theta'$ be the smallest congruence relations on $L$ with $L/\theta \in V$ and $L/\theta' \in V'$. Then by Theorem 2.3, $L \in K$ if and only if $\theta \cap \theta' \neq O_L$, the null congruence relation on $L$. In other words, $L \in K$ if and only if some nontrivial quotient in $L$ is collapsed by both $\theta$ and $\theta'$. The condition referred to can be expressed as follows.

$P_n(L)$ : There are nontrivial quotients $a/b$ and $c/d$ in $L$ such that

(i) $a/b \leq \alpha(\mu)/\beta(\mu)$ for some $\mu \in {}^{\omega}L$
(ii) $c/d \leq \gamma(\nu)/\delta(\nu)$ for some $\nu \in {}^{\omega}L$
(iii) $Pd(a/b, c/d) = n$.

By Collary 2.4, for any $L \in M$, $L \in K$ if and only if $P_n(L)$ holds for some natural number $n$. In other words, $K$ is defined, relative to $M$, by the disjunction of infinitely many formulas $P_n(L)$. We claim that if $P_n(L)$ holds for some $n \geq 19$, then $P_m(L)$ holds for some $m \leq 18$. Suppose $P_n(L)$ holds in $L$ for some $n \geq 19$. Then by the definition of $Pd(a/b, c/d) = n$, there exists a Hong's sequence $a'/b' = a_0/b_0, a_1/b_1, \cdots, a_n/b_n = c'/d'$ for some nontrivial subquotients $a'/b'$ of $a/b$ and $c'/d'$ of $c/d$, respectively. Also by Hong's Theorem, there exists the associated sequence of diamonds $D_1, D_2, \cdots, D_{n-1}$. Then we have the following two cases

(1) there exists a subsequence $D_k, D_{k+1}, D_{k+3}, D_{k+4}, D_{k+5}$ with $1 \le k \le n-3$ such that the sequence of numbers below the arrows is (2,1,2,2,2)

(2) there exists no such subsequences with $1 \le k \le n-3$.

Case(1) : By the construction of diamonds, $D_k \cup D_{k+1} \cup D_{k+3}$ forms a sublattice $L_0$ of which contains $A_1$ as a homomorphic image, and $a_{k+2}/b_{k+2}$ is a prime quotient in $L_0$. Since $A_1 \notin V$, $\beta(\overline{\mu'}) < \alpha(\overline{\mu'})$ for some $\mu' \in {}^{\omega}L_0$. Since $R(A_1) \le 7$, $\alpha(\overline{\mu'})/\beta(\overline{\mu'})$ projects onto $\bar{a}_{k+2}/\bar{b}_{k+2}$ in 7 steps. Therefore, by Lemma 3.2, a prime subquotient of $\alpha(\mu')/\beta(\mu')$ projects onto $a_{k+2}/b_{k+2}$ in 8 steps. Also, by Lemma 3.7, the lattice $\bar{E}$ is a homomorphic image of the sublattice $L_1$ of $L$ generated by $D_{k+2}, D_{k+3}, D_{k+4}, D_{k+5}$. Furthermore, $a_{k+2}/b_{k+2}$ is a prime quotient in $L_1$ and $\bar{a}_{k+2}/\bar{b}_{k+2}$ is a critical quotient in $\bar{E}$. Since $\bar{E} \notin V'$, $\delta(\overline{\nu'}) < \gamma(\overline{\nu'})$ for some $\nu' \in {}^{\omega}L_1$. Also, $R(\bar{E}) = 8$. Hence $\gamma(\overline{\nu'})/\delta(\overline{\nu'})$ projects onto $\bar{a}_{k+2}/\bar{b}_{k+2}$ in 8 steps. Since $a_{k+2}/b_{k+2}$ is a prime quotient in $L_1$, by Lemma 3.2, a prime subquotient of $\gamma(\nu')/\delta(\nu')$ projects onto $a_{k+2}/b_{k+2}$ in 9 steps. Thus a prime subquotient of $\alpha(\mu')/\beta(\mu')$ projects onto a prime subquotient of $\gamma(\nu')/\delta(\nu')$ in 17 steps. Therefore $P_m(L)$ holds for some $m < 18$.

Case(2) : By Hong's Theorem, we have two subcases.

(2.1) there exists a subsequence $D_k, D_{k+1}, D_{k+2}$ with $k = 1$ or 2 such that the sequence of numbers below the arrows is (2,2,2)

(2.2) there exists a subsequence $D_k, D_{k+1}, D_{k+2}$ with $n-5 \le k \le n-3$ such that the sequence of numbers below the arrows is (2,1,2).

Case(2.1) : By the argument of Case(1), a prime subquotient of $\gamma(\nu')/\delta(\nu')$ projects onto $a_k/b_k$ in 9 steps. Thus a prime subquotient of $\alpha(\mu)/\beta(\mu)$ projects onto a prime subquotient of $\gamma(\nu')/\delta(\nu')$ in $k+9$ steps. Since $k \le 2$, then $k+9 \le 11$. Therefore $P_m(L)$ holds for $m \le 11$.

Case(2.2) : By the argument of Case(1), a prime subquotient of $\alpha(\mu')/\beta(\mu')$ projects onto $a_{k+2}/b_{k+2}$ in 9 steps. Thus a prime subquotient of $\alpha(\mu')/\beta(\mu')$ projects onto a prime subquotient of $\gamma(\nu)/\delta(\nu)$ in $n-(k+2)+9$ steps. Since $n-5 \le k \le n-3$, then $10 \le n-k+7 \le 12$. Therefore $P_m(L)$ holds for $m \le 12$. The proof is complete.

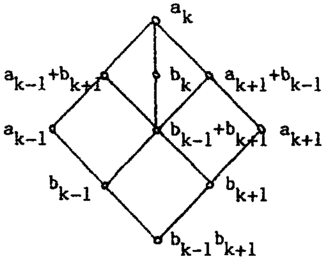
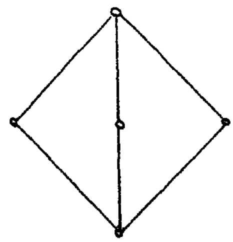
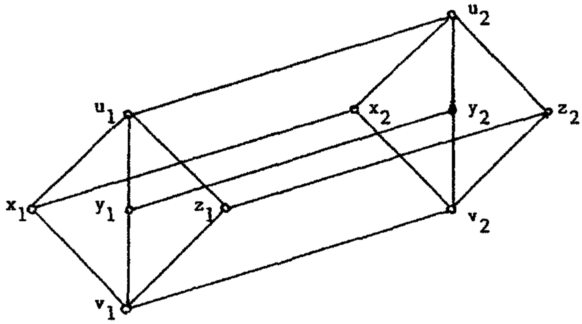



Figure 1



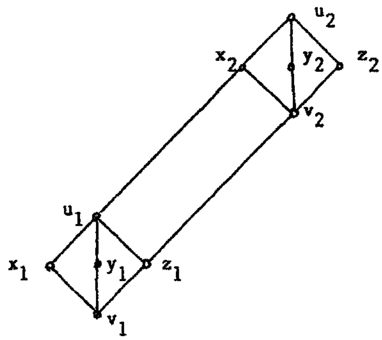
$M_3$

Figure 2



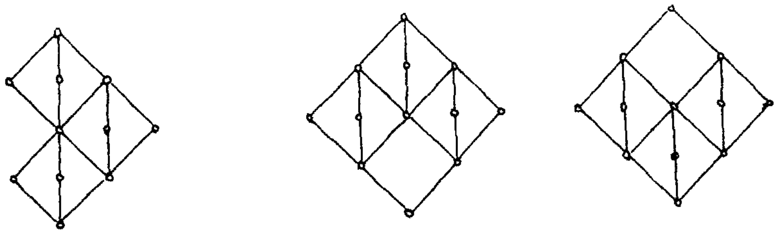

$D_1 \nearrow_{(1)} D_2$

Figure 3



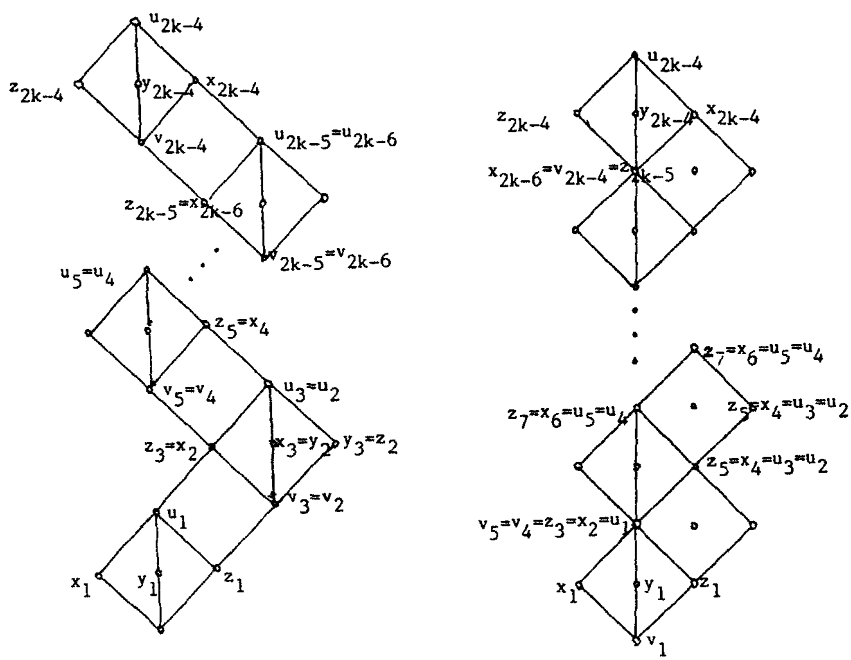

$D_1 \nearrow_{(2)} D_2$

Figure 4



$A_1$ $A_2$ $A_3$

Figure 5




$B_k'$ $B_k$

Figure 6

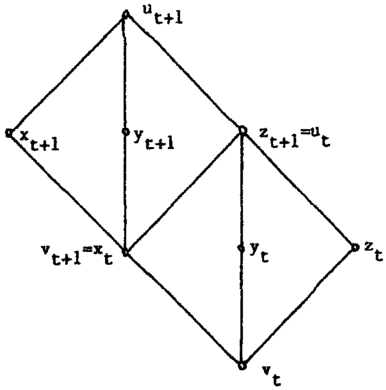
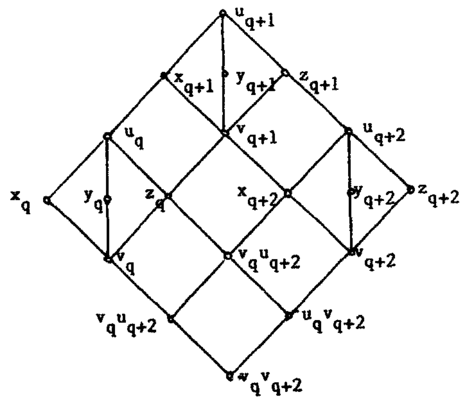
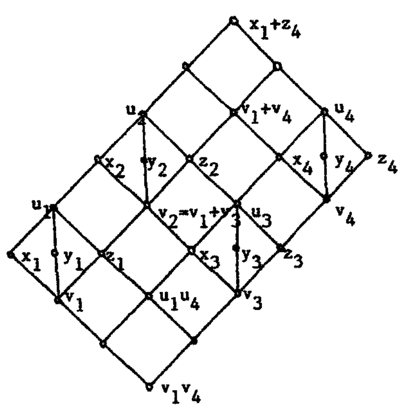



Figure 7



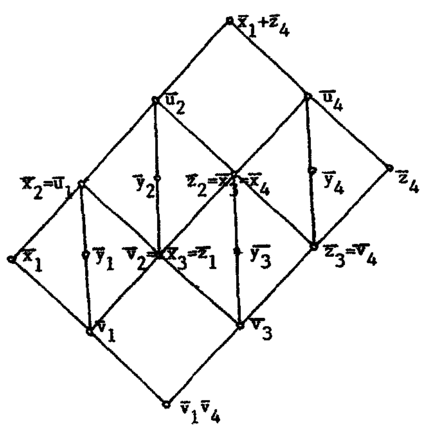

C

Figure 8




E

Figure 9




$\bar{E}$

Figure 10

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