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THE FEYNMAN INTEGRAL AND FEYNMAN'S OPERATIONAL CALCULUS: THE $\mathcal{L}(L_1(\mathbb{R}), C_0(\mathbb{R}))$ THEORY

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1. Introduction and Preliminaries

We begin by giving some of the notation which we will need. Let $\mathbb{R}$, $\mathbb{C}$, $\mathbb{C}^+$, and $\tilde{\mathbb{C}}^+$ denote the real numbers, the complex numbers, the complex numbers with positive real part, and the nonzero complex numbers with nonnegative real part, respectively. $L_p(\mathbb{R})(1 \le p < \infty)$ will denote the measurable, $\mathbb{C}$-valued functions on $\mathbb{R}$ which are pth power integrable. $C_0(\mathbb{R})$ will denote the $\mathbb{C}$-valued continuous functions on $\mathbb{R}$ which vanish at ∞ whereas $C[a,b]$ will denote the $\mathbb{R}$-valued continuous functions on $[a,b]$, and Wiener space, $C_0[a,b]$, will consist of those x in $C[a,b]$ such that $x(a) = 0$. Integration over $C_0[a,b]$ will always be with respect to Wiener measure m_w. $L_\infty(\mathbb{R})$ will denote the measurable, $\mathbb{C}$-valued functions on $\mathbb{R}$ which are essentially bounded. More formally, the elements of $L_p(\mathbb{R})$ and $L_\infty(\mathbb{R})$ are equivalence classes of functions, with Ψ_1 and Ψ_2 said to be equivalent if they are equal almost everywhere (a.e.) with respect to Lebesgue measure. $\mathcal{L}(L_1(\mathbb{R}), C_0(\mathbb{R}))$ will denote the space of bounded linear operators from $L_1(\mathbb{R})$ to $C_0(\mathbb{R})$.

In [2,3] Cameron and Storvick established the existence of an operator valued function space integral as an operator from $L_1(\mathbb{R})$ to $L_\infty(\mathbb{R})$ for certain functionals. In [7] the second author and Skoug extended the work [2] and obtained an existence theorem for the "Feynman integral" in the setting of the $\mathcal{L}(L_1(\mathbb{R}), C_0(\mathbb{R}))$ operator valued function space integral for more general functions. Furthermore, in [4] the first author obtained stability results for the "Feynman integral" in this setting.

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More recently [6], the second author and Lapidus established an existence theorem for the Feynman integral as a bounded linear operator on $L_2(\mathbb{R}^N)$ where a general finite Borel measure η on (a,b) is involved in the functionals rather than the usual Lebesgue measure ($y \mapsto f(\int_{(a,b)} \theta(s,y(s))d\eta(s))$ in place of $y \mapsto f(\int_{(a,b)} \theta(s,y(s))ds)$, for example). The theory in [6] is also used to make sense of Feynman's time-ordered operational calculus for noncommuting operators in a specific setting natural for nonrelativistic quantum mechanics. In [6], heavy use is made of the fact that both the space of functionals involved and the space of operators, $\mathcal{L}(L_2(\mathbb{R}^N))$, are Banach algebras. One of the interesting consequences of this work is that the Feynman integral can be used to make sense of some aspects of Feynman's operational calculus apart from the Banach algebra setting; specifically, in the setting of the $\mathcal{L}(L_1(\mathbb{R}), C_0(\mathbb{R}))$ Feynman integral.

We have not set out in this paper to give a complete development but rather to indicate what is possible. We will discuss at the end of the last section some ways in which our results can most probably be extended. Various comments about Feynman's operational calculus will be made, but we refer the reader to [6] for a more detailed discussion.

Some of our proofs are not too much different from proofs already in the literature. Where this occurs, we tend to be brief and to give appropriate references. The proof of Theorem 2.1, on the other hand, is not easy and some of the difficulties involved do not seem to be treated elsewhere in the literature.

We finish the present section by providing the additional definitions and facts which we will require.

$\mathcal{M}(a,b)$ will denote the space of complex Borel measures η on the open interval (a,b). $\mathcal{M}(a,b)$ is a Banach space under the total variation norm and the natural operations. A measure μ in $\mathcal{M}(a,b)$ is said to be continuous if $\mu(\{\tau\}) = 0$ for every τ in (a,b). ν in $\mathcal{M}(a,b)$ is discrete (or is a "pure point measure" in the terminology of Reed and Simon [9]) if and only if there is an at most countable subset $\{\tau_p\}$ of (a,b) and a summable sequence $\{\omega_p\}$ from $\mathbb{C}$ such that

$$\mu = \sum_{p=1}^{\infty} \omega_p \delta_{\tau_p}$$

where δ_{τ_p} is the Dirac measure with total mass one concentrated at τ_p. Every measure $\eta \in \mathcal{M}(a,b)$ has a unique decomposition, $\eta = \mu + \nu$, into a continuous part μ and a discrete part ν [9, p.22]. We work with the space $\mathcal{M}(a,b)$ throughout, but $\mathcal{M}[a,b]$ could be treated without any essential complications. However, allowing η to have nonzero mass at a or b introduces additional alternatives which we have chosen to avoid.

Let $F : C[a,b] \to \mathbb{C}$ be a functional. Given $\lambda > 0$, $\Psi \in L_1(\mathbb{R})$ and ξ in $\mathbb{R}$, let

$$(1.1) \quad (I_\lambda(F)\Psi)(\xi) = \int_{C_0[a,b]} F(\lambda^{-\frac{1}{2}}x + \xi)\Psi(\lambda^{-\frac{1}{2}}x(b) + \xi)dm_w(x).$$

If $I_\lambda(F)\Psi$ is in $C_0(\mathbb{R})$ as a function of ξ and if the correspondence $\Psi \to I_\lambda(F)\Psi$ gives an element of $\mathcal{L} \equiv \mathcal{L}(L_1(\mathbb{R}), C_0(\mathbb{R}))$, we say that the operator-valued function space integral $I_\lambda(F)$ exists.

Next suppose there exist $\lambda_0\,(0 < \lambda_0 \leq \infty)$ such that $I_\lambda(F)$ exists for all $\lambda \in (0, \lambda_0)$ and further suppose that there exists an $\mathcal{L}$-valued function which is analytic in $\mathbb{C}^+_{\lambda_0} \equiv \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}\ \lambda > 0,\ |\lambda| < \lambda_0\}$ and agrees with $I_\lambda(F)$ on $(0, \lambda_0)$; then this $\mathcal{L}$-valued function is denoted by $I^{an}_\lambda(F)$ and is called the operator-valued function space integral of F associated with λ.

Finally, let q be in $\mathbb{R}$ with $|q| < \lambda_0$. Suppose there exists an operator $J^{an}_q(F)$ in $\mathcal{L}$ such that for every Ψ in $L_1(\mathbb{R})$, $J^{an}_q(F)\Psi$ is the weak limit of $I^{an}_\lambda(F)\Psi$ as $\lambda \to -iq$ through $\mathbb{C}^+_{\lambda_0}$; then $J^{an}_q(F)$ is called the operator-valued Feynman integral (or Cameron-Storvick function space integral) of F associated with $-iq$.

The above definitions follow those of the second author and Skoug [7] and are more stringent than those of Cameron and Storvick [2]. Where we have $C_0(\mathbb{R})$, they have $L_\infty(\mathbb{R})$ in [2]. Also their "weak limits" and analyticity are based on the weak* topology on $L_\infty(\mathbb{R})$ induced by its pre-dual $L_1(\mathbb{R})$; our weak limits and analyticity are based on the weak topology on $C_0(\mathbb{R})$ induced by its dual.

For $s > 0$, $\lambda \in \tilde{\mathbb{C}}^+$, and $u \in \mathbb{R}$, let

$$(1.2) \qquad e_{s/\lambda}(u) := (\lambda/2\pi s)^{\frac{1}{2}} \exp\left\{-\frac{\lambda u^2}{2s}\right\}.$$

Further, let $C_{s/\lambda}$ be the operator of convolution by $e_{s/\lambda}$; that is, for Ψ in $L_1(\mathbb{R})$ and ξ in $\mathbb{R}$,

$$(C_{s/\lambda}\Psi)(\xi) := \int_{\mathbb{R}} \Psi(u)e_{s/\lambda}(\xi - u)du. \tag{1.3}$$

Using the Lebesgue Dominated Convergence Theorem and the Riemann-Lebesgue Theorem, $C_{s/\lambda}\Psi$ is in $C_0(\mathbb{R})$ and so $C_{s/\lambda}$ is in $\mathcal{L}(L_1(\mathbb{R}), C_0(\mathbb{R}))$. It is easy to see that

$$\|C_{s/\lambda}\| \leq (|\lambda|/2\pi s)^{\frac{1}{2}}. \tag{1.4}$$

If f is in $C_0(\mathbb{R})$ and g is in $L_1(\mathbb{R})$, then fg is in $L_1(\mathbb{R})$ and

$$\|fg\|_1 \leq \|f\|_\infty \|g\|_1. \tag{1.5}$$

Let M_g be the operator of multiplication by g. Then M_g is in $\mathcal{L}(C_0(\mathbb{R}), L_1(\mathbb{R}))$ and

$$\|M_g\| \leq \|g\|_1. \tag{1.6}$$

Let h be in $L_\infty(\mathbb{R}) \cap L_1(\mathbb{R})$ and let s be a positive integer. Then the operator M_{h^s} is in $\mathcal{L}(C_0(\mathbb{R}), L_1(\mathbb{R}))$ and we have

$$\|M_{h^s}(\beta)\|_1 = \|h^s\beta\|_1 \leq \|h\|_\infty^{S-1}\|h\|_1\|\beta\|_\infty \tag{1.7}$$

for $\beta \in C_0(\mathbb{R}) \subset L_\infty(\mathbb{R})$.

2. The Generating Functionals

We will consider in the next section functionals which map $y \in C[a,b]$ into $f(\int_{(a,b)} \theta(s, y(s))d\eta(s))$ for certain analytic functions f. However, the hardest work comes in this section where f is just the nth power function.

Let η be in $\mathcal{M}(a,b)$. Then η has a canonical representation [9, p.23; 6, p.24] $\eta = \nu + \mu + \eta_{sc}$ where ν is discrete, μ is absolutely continuous with respect to Lebesgue measure m_L, and η_{sc} is continuous and singular relative to Lebesgue measure. Although more generality seems possible as we will discuss in our closing remarks, we will assume throughout the paper that $\eta_{sc}=0$ and that ν is finitely supported. In this case, $\nu=\sum_{j=1}^{m}\omega_j\delta_{\tau_j}$ and $\eta=\sum_{j=1}^{m}\omega_j\delta_{\tau_j}+\mu$ where δ_{τ_j} is the Dirac measure at $\tau_j\in(a,b)$, $a<\tau_1<\cdots<\tau_m<b$, and $\omega_j\in\mathbb{C}$ for $j=1,\cdots,m$.

The calculation involved in the first lemma will be important to us as we continue. We give an example following the proof which illustrates the somewhat complicated notation. The reader may find it useful to check the example both before and after reading the proof.

The time-ordering involved in the definition of the sets $\Delta^{z_1,\cdots,z_{m-k}}_{q_0;j_1,\cdots,j_{m-k+1}}$ below will help us to calculate certain Wiener integrals and will be reflected in the time-ordering of the noncommuting operators in Theorem 2.1. Time-ordering is an essential feature of Feynman's operational calculus (see [6]).

LEMMA 2.1. *For any nonnegative integer $q_0, r>2$, and r' satisfying $1/r+1/r'=1$, we have, for $k=0,1,\cdots,m-1$,*

$$\begin{aligned}(2.1)\qquad & A(q_0;\tau_{z_1},\cdots,\tau_{z_{m-k}};r') \\ &:=\Big\{\sum_{j_1+\cdots+j_{m-k+1}=q_0}\int_{\Delta^{z_1,\cdots,z_{m-k}}_{q_0;j_1,\cdots,j_{m-k+1}}}[(s_1-a)\cdots(\tau_{z_1}-s_{j_1}) \\ &\qquad(s_{j_1+1}-\tau_{z_1})\cdots(\tau_{z_{m-k}}-s_{j_1+\cdots+j_{m-k}}) \\ &\qquad(s_{j_1+\cdots+j_{m-k}+1}-\tau_{z_{m-k}})\cdots(b-s_{q_0})]^{-\frac{r'}{2}}X_{i=1}^{q_0}ds_i\Big\}^{\frac{1}{r'}} \\ &=\Big\{\sum_{j_1+\cdots+j_{m-k+1}=q_0}\Big[\prod_{l=1}^{m-k+1}\Big[(\tau_{z_l}-\tau_{z_{l-1}})^{j_l-(j_l+1)r'/2} \\ &\qquad\frac{[\Gamma(1-r'/2)]^{j_l+1}}{\Gamma[(j_l+1)(1-r'/2)]}\Big]\Big]\Big\}^{1/r'}\end{aligned}$$

where $a=\tau_{z_0}\le\tau_{z_1}<\cdots<\tau_{z_{m-k}}\le\tau_{z_{m-k+1}}=b$, Γ denotes the gamma function, $j_1,\cdots,j_{m-k+1}$, $z_1,\cdots,z_{m-k}$ are nonnegative inte-

gers, and

$$\begin{aligned}(2.2)\qquad &\Delta^{z_1,\cdots,z_{m-k}}_{q_0;j_1,\cdots,j_{m-k+1}}\\ &= \{(s_1,\cdots,s_{q_0})\in(a,b)^{q_0}\,|\,a<s_1<\cdots\\ &\quad< s_{j_1}<\tau_{z_1}<s_{j_1+1}<\cdots<s_{j_1+\cdots+j_{m-k}}<\tau_{z_{m-k}}\\ &\quad< s_{j_1+\cdots+j_{m-k}+1}<\cdots<s_{q_0}<b\}.\end{aligned}$$

Note that for $q_0=0$ there are no s's to be integrated out and for $k=0,1,\cdots,m-k$,

$$(2.3)\quad A(0;\tau_{z_1},\cdots,\tau_{z_{m-k}};r') = [(\tau_{z_1}-a)(\tau_{z_2}-\tau_{z_1})\cdots(b-\tau_{z_{m-k}})]^{-\frac{r'}{2}}.$$

Furthermore, if there are no τ's in (a,b), that is, if $k=m$, then we have

$$\begin{aligned}(2.4)\quad &A(q_0;\cdot;r')\\ &:= \left\{\int_{\Delta_{q_0}}[(s_1-a)(s_2-s_1)\cdots(b-s_{q_0})]^{-r'/2}ds_1\cdots ds_{q_0}\right\}^{1/r'}\\ &= \left\{(b-a)^{q_0-(q_0+1)r'/2}\frac{[\Gamma(1-r'/2)]^{q_0+1}}{\Gamma[(q_0+1)(1-r'/2)]}\right\}^{1/r'}\end{aligned}$$

where

$$(2.5)\qquad \Delta_{q_0} = \{(s_1,\cdots,s_{q_0})\in(a,b)^{q_0}\,|\,a<s_1<\cdots<s_{q_0}<b\}.$$

Proof. Using the Dirichlet integral [10; p.258], we have, for $r>2$ and for $0<s_1<\cdots<s_{j_k}<\tau_{z_k}-\tau_{z_{k-1}}$, $k=1,\cdots,m+1$,

$$\begin{aligned}(2.6)\quad &\int_0^{\tau_{z_k}-\tau_{z_{k-1}}}\int_0^{s_{j_k}}\cdots\int_0^{s_2}[s_1(s_2-s_1)\cdots\\ &\qquad((\tau_{z_k}-\tau_{z_{k-1}})-s_{j_k})]^{-r'/2}ds_1\cdots ds_{j_k}\\ &= \begin{cases}(\tau_{z_k}-\tau_{z_{k-1}})^{j_k-(j_k+1)r'/2}\frac{[\Gamma(1-r'/2)]^{j_k+1}}{\Gamma[(j_k+1)(1-r'/2)]} & (j_k=1,2,\cdots,q_0)\\ (\tau_{z_k}-\tau_{z_{k-1}})^{-r'/2} & (j_k=0).\end{cases}\end{aligned}$$

From (2.2), the Fubini Theorem, and a simple change of variables, we obtain

(2.7)

$$\int_{\Delta^{z_1,\dots,z_{m-k}}_{q_0;j_1,\dots,j_{m-k+1}}} [(s_1-a)\cdots(\tau_{z_1}-s_{j_1})(s_{j_1+1}-\tau_{z_1})\cdots (b-s_{q_0})]^{-r'/2} X_{i=1}^{q_0} ds_i$$

$$= \left\{\int_a^{\tau_{z_1}}\int_a^{s_{j_1}}\cdots\int_a^{s_2}[(s_1-a)(s_2-s_1)\cdots(\tau_{z_1}-s_{j_1})]^{-r'/2}ds_1\cdots ds_{j_1}\right\}$$

$$\times\left\{\int_{\tau_{z_1}}^{\tau_{z_2}}\int_{\tau_{z_1}}^{s_{j_1+j_2}}\cdots\int_{\tau_{z_1}}^{s_{j_1+2}}[(s_{j_1+1}-\tau_{z_1})(s_{j_1+2}-s_{j_1+1}) \cdots(\tau_{z_2}-s_{j_1+j_2})]^{-r'/2}ds_{j_1+1}\cdots ds_{j_1+j_2}\right\}\times\cdots\times$$

$$\left\{\int_{\tau_{z_{m-k}}}^{b}\int_{\tau_{z_{m-k}}}^{s_{q_0}}\cdots\int_{\tau_{z_{m-k}}}^{s_{j_1+\cdots+j_{m-k}+2}}[(s_{j_1+\cdots+j_{m-k+1}+1}-\tau_{z_{m-k}})\cdots (b-s_{q_0})]^{-r'/2}ds_{j_1+\cdots+j_{m-k}+1}\cdots ds_{q_0}\right\}$$

$$= \left\{\int_0^{\tau_{z_1}-a}\int_0^{s_{j_1}}\cdots\int_0^{s_{j_2}}[s_1(s_2-s_1)\cdots(\tau_{z_1}-a)-s_{j_1})]^{-r'/2}ds_1\cdots ds_{j_1}\right\}$$

$$\times\left\{\int_0^{\tau_{z_2}-\tau_{z_1}}\int_0^{s_{j_2}}\cdots\int_0^{s_2}[s_1(s_2-s_1)\cdots (\tau_{z_2}-\tau_{z_1})-s_{j_2})]^{-r'/2}ds_1\cdots ds_{j_2}\right\}\times\cdots$$

$$\times\left\{\int_0^{b-\tau_{z_{m-k}}}\int_0^{s_{j_{m-k+1}}}\cdots\int_0^{s_2}[s_1(s_2-s_1)\cdots ((b-\tau_{z_{m-k}})-s_{j_{m-k+1}})]^{-r'/2}ds_1\cdots ds_{m-k+1}\right\}.$$

From (2.6) and (2.7) we have our lemma.

The following specific example illustrates the notation and the conclusion of the preceding lemma:

$$\Delta^{1,2,3}_{4;1,2,0,1} = \{(s_1, \cdots, s_4) \in (a,b)^4 \mid a < s_1 < \tau_1 < s_2 < s_3 < \tau_2 < \tau_3 < s_4 < b\}$$

and then

$$\int_{\Delta^{1,2,3}_{4;1,2,0,1}} [(s_1 - a)(\tau_1 - s_1)\cdots(s_4 - \tau_3)(b - s_4)]^{-\frac{r'}{2}} X^4_{i=1} ds_i$$
$$= (\tau_1 - a)^{1-r'}(\tau_2 - \tau_1)^{\frac{4-3r'}{2}}(\tau_3 - \tau_2)^{-\frac{r'}{2}}(b - \tau_3)^{1-r'} \frac{[\Gamma(1-\frac{r'}{2})]^7}{[\Gamma(2-r')]^2\Gamma(3-\frac{3}{2}r')}.$$

As we continue we will need to write

$$[\omega_1\theta(\tau_1, y(\tau_1)) + \cdots + \omega_m\theta(\tau_m, y(\tau_m)) + \theta(s, y(s))]^n$$

as a product of monomials. Of course, the multinomial formula would do this for us. However, we will need to know in each term precisely which subset of $\{\tau_1, \cdots, \tau_m\}$ actually appears and so we will need a more refined breakdown of the sum. It will be convenient to introduce a primed notation on sums like $\sum'_{q_0+q_1+\cdots+q_{m-k}=n}$; this sum is to be over integers $q_0, q_1, \cdots, q_{m-k}$ where $q_0 \geq 0, q_1 \geq 1, \cdots, q_{m-k} \geq 1$, and, of course, $q_0 + q_1 + \cdots + q_{m-k} = n$. Using this notation, we can write

(2.8)
$$[\omega_1\theta(\tau_1, y(\tau_1)) + \cdots + \omega_m\theta(r_m, y(\tau_m)) + \theta(s, y(s))]^n$$
$$= \sum'_{q_0+q_1+\cdots+q_m=n} \frac{n!}{q_0!q_1!\cdots q_m!}[\omega_1\theta(\tau_1, y(\tau_1))]^{q_1}\cdots$$
$$[\omega_m\theta(\tau_m, y(\tau_m))]^{q_m}[\theta(s, y(s))]^{q_0}$$
$$+ \sum_{1\leq z_1<\cdots<z_{m-1}\leq m} \sum'_{q_0+q_1+\cdots+q_{m-1}=n} \frac{n!}{q_0!\cdots q_{m-1}!}[\omega_{z_1}\theta(\tau_{z_1}, y(\tau_{z_1}))]^{q_1}$$
$$\cdots[\omega_{z_{m-1}}\theta(\tau_{z_{m-1}}, y(\tau_{z_{m-1}}))]^{q_{m-1}}[\theta(s, y(s))]^{q_0}$$
$$+ \sum_{1\leq z_1<\cdots<z_{m-2}\leq m} \sum'_{q_0+q_1+\cdots+q_{m-2}=n} \frac{n!}{q_0!\cdots q_{m-2}!}[\omega_{z_1}\theta(\tau_{z_1}, y(\tau_{z_1}))]^{q_1}$$

$$\cdots [\omega_{z_{m-2}}\theta(\tau_{z_{m-2}}, y(\tau_{z_{m-2}}))]^{q_{m-2}}[\theta(s, y(s))]^{q_0} + \cdots$$

$$+ \sum_{1\le z_1<\cdots<z_{m-k}\le m}\ \sum_{q_0+q_1+\cdots+q_{m-k}=n}{}' \frac{n!}{q_0!\cdots q_{m-k}!}[\omega_{z_1}\theta(\tau_{z_1}, y(\tau_{z_1}))]^{q_1}$$

$$\cdots [\omega_{z_{m-k}}\theta(\tau_{z_{m-k}}, y(\tau_{z_{m-k}}))]^{q_{m-k}}[\theta(s, y(s))]^{q_0} + \cdots$$

$$+ \sum_{1\le z_1\le m}\ \sum_{q_0+q_1=n}{}' \frac{n!}{q_0!q_1!}[\omega_{z_1}\theta(\tau_{z_1}, y(\tau_{z_1}))]^{q_1}[\theta(s, y(s))]^{q_0}$$

$$+ [\theta(s, y(s))]^n.$$

We rewrite this expression in more compact form in our second lemma. Note that when $k = m$, the inner sums below collapse to the single term $[\theta(s, y(s))]^n$.

LEMMA 2.2.

$$\Big[\sum_{i=1}^{m}\omega_i\theta(\tau_i, y(\tau_i)) + \theta(s, y(s))\Big]^n \tag{2.9}$$

$$= \sum_{k=0}^{m}\ \sum_{1\le z_1<\cdots<z_{m-k}\le m}\ \sum_{q_0+q_1+\cdots+q_{m-k}=n}{}' \frac{n!}{q_0!q_1!\cdots q_{m-k}!}$$

$$[\omega_{z_1}\theta(\tau_{z_1}, y(\tau_{z_1}))]^{q_1}$$

$$\cdots [\omega_{z_{m-k}}\theta(\tau_{z_{m-k}}, y(\tau_{z_{m-k}}))]^{q_{m-k}}[\theta(s, y(s))]^{q_0}.$$

The formula in the following lemma is like (2.9) except that $\theta(s, y(s))$ is not involved at all. In this case the prime on the sum over q's in (2.10) is intended to mean that none of the q's involved is zero, that is, $q_k \ge 1$ for $k \ge 1$.

LEMMA 2.3.

$$\Big[\sum_{i=1}^{m}\omega_i\theta(\tau_i, y(\tau_i))\Big]^n \tag{2.10}$$

$$= \sum_{k=0}^{m-1}\ \sum_{1\le z_1<\cdots<z_{m-k}\le m}\ \sum_{q_1+\cdots+q_{m-k}=n}{}' \frac{n!}{q_1!\cdots q_{m-k}!}$$

$$[\omega_{z_1}\theta(\tau_{z_1}, y(\tau_{z_1}))]^{q_1}\cdots [\omega_{z_{m-k}}\theta(\tau_{z_{m-k}}, y(\tau_{z_{m-k}}))]^{q_{m-k}}.$$

THEOREM 2.1. *For each nonnegative integer n and $y \in C[a,b]$, let*

$$F_n(y) := \left[\int_{(a,b)} \theta(s, y(s))d\eta(s)\right]^n \tag{2.11}$$

where $\eta = \mu + \sum_{j=1}^{m} \omega_j \delta_{\tau_j}$ with μ absolutely continuous with respect to Lebesgue measure; further suppose that for fixed $r > 2$, θ is a $\mathbb{C}$-valued function on $(a,b) \times \mathbb{R}$ satisfying the following 2 conditions:

$$\phi(s) := \|\theta(s,\cdot)\|_1 \frac{d|\mu|}{dm_L}(s) \in L_r[a,b]. \tag{2.12a}$$

$$\theta(\tau_j, \cdot) \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R}) \text{ for each } j = 1, \cdots, m. \tag{2.12b}$$

Then the operators $I_\lambda^{an}(F_n)$ and $J_q^{an}(F_n)$ exist for all $\lambda \in \mathbb{C}^+$ and all real $q \neq 0$, respectively. Further for $\lambda \in \mathbb{C}^+$, $\Psi \in L_1(\mathbb{R})$, and $\xi \in \mathbb{R}$,

$$\begin{aligned}(2.13)\quad &(I_\lambda^{an}(F_n)\Psi)(\xi) \\ &= \Big\{\sum_{k=0}^{m} \sum_{1\le z_1<\cdots<z_{m-k}\le m} \sideset{}{'}\sum_{q_0+q_1+\cdots+q_{m-k}=n} \frac{n!\omega_{z_1}^{q_1}\cdots\omega_{z_{m-k}}^{q_{m-k}}}{q_1!\cdots q_{m-k}!} \\ &\qquad \Big[\sum_{j_1+\cdots+j_{m-k+1}=q_0} \int_{\Delta_{q_0;j_1,\cdots,j_{m-k+1}}^{z_1,\cdots,z_{m-k}}} L_0 \circ L_1 \circ \cdots \\ &\qquad\qquad \circ L_{m-k} X_{i=1}^{q_0} d\mu(s_i)\Big]\Psi\Big\}(\xi)\end{aligned}$$

where $\Delta_{q_0;j_1,\cdots,j_{m-k+1}}^{z_1,\cdots,z_{m-k}}$ is given by (2.2) and, for $(s_1, \cdots, s_{q_0}) \in \Delta_{q_0;j_1,\cdots,j_{m-k+1}}^{z_1,\cdots,z_{m-k}}$ and $\alpha \in \{0, 1, \cdots, m-k\}$,

$$\begin{aligned}(2.14)\quad L_\alpha = \theta(\tau_{z_\alpha})^{q_\alpha} &\circ C_{(s_{j_1+\cdots+j_\alpha+1} - \tau_{z_\alpha})/\lambda} \circ \theta(s_{j_1+\cdots+j_\alpha+1}) \\ &\circ \cdots \circ \theta(s_{j_1+\cdots+j_{\alpha+1}}) \circ C_{(\tau_{z_{\alpha+1}} - s_{j_1+\cdots+j_{\alpha+1}})/\lambda}.\end{aligned}$$

(It is convenient to let $\theta(\tau)^q$ denote the operator of multiplication by $[\theta(\tau,\cdot)]^q$, that is, $\theta(\tau)^q = M_{[\theta(\tau,\cdot)]^q}$. We use the conventions $\tau_0 = a$, $\tau_{m+1} = b$, and $\theta(\tau_0)^{q_0} = 1$.)

For real $q \neq 0$, $(J_q^{an}(F_n)\Psi)(\xi)$ is given by the right-hand side of (2.13) with $\lambda = -iq$. Finally we have for $\lambda \in \mathbb{C}^+$,

$$(2.15)\|I_\lambda^{an}(F_n)\| \le b_n(|\lambda|)$$

$$:= \sum_{k=0}^{m} \sum_{1\le z_1<\cdots<z_{m-k}\le m} \sum_{q_0+q_1+\cdots+q_{m-k}=n}{}' \left[\frac{|\lambda|}{2\pi}\right]^{(q_0+m-k+1)/2}$$

$$\frac{n!|\omega_{z_1}|^{q_1}\cdots|\omega_{z_{m-k}}|^{q_{m-k}}}{q_1!\cdots q_{m-k}!}\left[\frac{(q_0+m-k)!}{(m-k)!}\right]^{\frac{1}{2}}(q_0!)^{-(2+r)/2r}\|\phi\|_r^{q_0}$$

$$\left[\prod_{l=1}^{m-k}\|\theta(\tau_{z_l},\cdot)\|_\infty^{q_l-1}\|\theta(\tau_{z_l},\cdot)\|_1\right]A(q_0;\tau_{z_1},\cdots,\tau_{z_{m-k}};r')$$

where $A(q_0;\tau_{z_1},\cdots,\tau_{z_{m-k}};r')$ is given by (2.1) (or by (2.4) when $k = m$). The bound (2.15) also holds for $J_q^{an}(F_n)$ with $|\lambda|$ replaced by $|q|$.

Note. The ordering of the noncommuting operators appearing in (2.13) and (2.14) corresponds to the time ordering of the indices involved. Thus the "disentangling process" which is the key to Feynman's operational calculus (see the discussion in [6]) is brought about naturally by the function space integrals $I_\lambda^{an}(F_n)$ and $J_q^{an}(F_n)$.

Proof. As mentioned in the introduction, this will be our most difficult proof. Let $\Psi \in L_1(\mathbb{R})$, $\xi \in \mathbb{R}$ and $\lambda > 0$ be given. Then by (1.1), (2.11), and using Lemma 2.2,

(2.16)

$$(I_\lambda(F_n)\Psi)(\xi)$$

$$= \int_{C_0[a,b]} F_n(\lambda^{-\frac{1}{2}}x+\xi)\Psi(\lambda^{-\frac{1}{2}}x(b)+\xi)dm_w(x)$$

$$= \int_{C_0[a,b]} \left[\sum_{j=1}^{m}\omega_j\theta(\tau_j,\lambda^{-\frac{1}{2}}x(\tau_j)+\xi)+\int_{(a,b)}\theta(s,\lambda^{-\frac{1}{2}}x(s)+\xi)d\mu(s)\right]^n$$

$$\Psi(\lambda^{-\frac{1}{2}}x(b)+\xi)dm_w(x)$$

$$= \int_{C_0[a,b]} \sum_{k=0}^{m} \sum_{1\le z_1<\cdots<z_{m-k}\le m} \sum_{q_0+q_1+\cdots+q_{m-k}=n}{}' \frac{n!}{q_0!q_1!\cdots q_{m-k}!}$$

$$[\int_{(a,b)} \theta(s, \lambda^{-\frac{1}{2}}x(s)+\xi)d\mu(s)]^{q_0} \prod_{l=1}^{m-k} \left[\omega_{z_l}\theta(\tau_{z_l}, \lambda^{-\frac{1}{2}}x(\tau_{z_l})+\xi)\right]^{q_l}$$
$$\Psi(\lambda^{-\frac{1}{2}}x(b)+\xi)dm_w(x).$$

Using the simplex trick, we have by (2.5) and (2.16)

$$(2.17)\ (I_\lambda(F_n)\Psi)(\xi)$$
$$= \int_{C_0[a,b]} \sum_{k=0}^{m} \sum_{1\le z_1<\cdots<z_{m-k}\le m} \sum\nolimits'_{q_0+q_1+\cdots+q_{m-k}=n} \frac{n!q_0!}{q_0!q_1!\cdots q_{m-k}!}$$
$$\left[\int_{\Delta_{q_0}} \prod_{i=1}^{q_0} \theta(s_i, \lambda^{-\frac{1}{2}}x(s_i)+\xi) X_{i=1}^{q_0} d\mu(s_i)\right]$$
$$\prod_{l=1}^{m-k} [\omega_{z_l}\theta(\tau_{z_l}, \lambda^{-\frac{1}{2}}x(\tau_{z_l})+\xi)]^{q_l}\Psi(\lambda^{-\frac{1}{2}}x(b)+\xi)dm_w(x)$$
$$= \sum_{k=0}^{m} \sum_{1\le z_1<\cdots<z_{m-k}\le m} \sum\nolimits'_{q_0+q_1+\cdots+q_{m-k}=n} \frac{n!\omega_{z_1}^{q_1}\cdots\omega_{z_{m-k}}^{q_{m-k}}}{q_1!\cdots q_{m-k}!}$$
$$\left[\sum_{j_1+\cdots+j_{m-k+1}=q_0} \int_{\Delta_{q_0;j_1,\cdots,j_{m-k+1}}^{z_1,\cdots,z_{m-k}}} Y X_{i=1}^{q_0} d\mu(s_i)\right]$$

where

$$(2.18)\quad Y := \int_{C_0[a,b]} \left[\prod_{i=1}^{q_0} \theta(s_i, \lambda^{-\frac{1}{2}}x(s_i)+\xi)\right]$$
$$\left[\prod_{l=1}^{m-k} (\theta(\tau_{z_l}, \lambda^{-\frac{1}{2}}x(\tau_{z_l})+\xi))^{q_l}\right] \Psi(\lambda^{-\frac{1}{2}}x(b)+\xi)dm_w(x).$$

The last equality in (2.17) comes from the Fubini Theorem which will be justified later in conjunction with the norm estimate (2.15). Using the basic Wiener integration formula and a simple change of variables, we have

(2.19)

$$
\begin{aligned}
Y = &\Big[(2\pi)^{q_0+m-k+1}(s_1-a)\cdots(\tau_{z_1}-s_{z_1})(s_{j_1+1}-\tau_{z_1})\cdots \\
&(b-s_{q_0})\Big]^{-\frac{1}{2}}\int_{\mathbb{R}^{q_0+m-k+1}}\Big[\prod_{i=1}^{q_0}\theta(s_i,\lambda^{-\frac{1}{2}}u_i+\xi)\Big] \\
&\Big[\prod_{l=1}^{m-k}(\theta(\tau_{z_l},\lambda^{-\frac{1}{2}}u_l'+\xi))^{q_l}\Big]\Psi(\lambda^{-\frac{1}{2}}u_{m-k+1}'+\xi) \\
&\exp\Big\{-\frac{(u_1-u_0)^2}{2(s_1-a)}-\cdots-\frac{(u_1'-u_{j_1})^2}{2(\tau_{z_1}-s_{j_1})}-\frac{(u_{j_1+1}-u_1')^2}{2(s_{j_1+1}-\tau_{z_1})}-\cdots \\
&-\frac{(u_{m-k+1}'-u_{q_0})^2}{2(b-s_{q_0})}\Big\}\Big[X_{i=1}^{q_0}du_i\Big]\Big[X_{j=1}^{m-k+1}du_j'\Big] \\
= &(\lambda/2\pi)^{(q_0+m-k+1)/2}[(s_1-a)\cdots(\tau_{z_1}-s_{j_1})(s_{j_1+1}-\tau_{z_1})\cdots \\
&(b-s_{q_0})]^{-\frac{1}{2}}\int_{\mathbb{R}^{q_0+m-k+1}}\Big[\prod_{i=1}^{q_0}\theta(s_i,v_i)\Big] \\
&\Big[\prod_{l=1}^{m-k}(\theta(\tau_{z_l},v_l'))^{q_l}\Big]\Psi(v_{m-k+1}') \\
&\exp\Big\{-\frac{\lambda(v_1-\xi)^2}{2(s_1-a)}-\cdots-\frac{\lambda(v_1'-v_{j_1})^2}{2(\tau_{z_1}-s_{j_1})}-\frac{\lambda(v_{j_1+1}-v_1')^2}{2(s_{j_1+1}-\tau_{z_1})}-\cdots \\
&-\frac{\lambda(v_{m-k+1}'-v_{q_0})^2}{2(b-s_{q_0})}\Big\}\Big[X_{i=1}^{q_0}dv_i\Big]\Big[X_{j=1}^{m-k+1}dv_j'\Big], \quad (v_0\equiv\xi).
\end{aligned}
$$

Note that in the notation in (2.19) we have assumed for the sake of definiteness that there is at least one s in the interval (a,τ_{z_1}); actually this need not be so in which case s_1-a is replaced by $\tau_{z_1}-a$. A similar remark applies to the interval $(\tau_{z_{m-k}},b)$.

Using (1.2) and (1.3), we obtain by (2.19)

$$Y=((L_0\circ L_1\circ\cdots\circ L_{m-k})\Psi)(\xi) \tag{2.20}$$

where for $(s_1, \cdots, s_{q_0}) \in \Delta^{z_1,\cdots,z_{m-k}}_{q_0;j_1,\cdots,j_{m-k+1}}$ and $\alpha \in \{0, 1, \cdots, m-k\}$, L_α is given by (2.14).

From (2.13) for $\lambda > 0$, we have

$$
\begin{aligned}
(2.21)\qquad & \|I_\lambda(F_n)\| \\
&\le \sum_{k=0}^{m} \sum_{1\le z_1<\cdots<z_{m-k}\le m} \sum\nolimits'_{q_0+q_1+\cdots+q_{m-k}=n} \\
&\qquad \frac{n!|\omega_{z_1}|^{q_1}\cdots|\omega_{z_{m-k}}|^{q_{m-k}}}{q_1!\cdots q_{m-k}!} B(q_0;\tau_{z_1},\cdots,\tau_{z_{m-k}};\mu)
\end{aligned}
$$

where

$$
\begin{aligned}
(2.22)\qquad & B(q_0;\tau_{z_1},\cdots,\tau_{z_{m-k}};\mu) \\
&:= \sum_{j_1+\cdots+j_{m-k+1}=q_0} \int_{\Delta^{z_1,\cdots,z_{m-k}}_{q_0;j_1,\cdots,j_{m-k+1}}} \\
&\qquad \|L_0\circ L_1\circ\cdots\circ L_{m-k}\| X^{q_0}_{i=1} d|\mu|(s_i).
\end{aligned}
$$

Using the norm inequalities (1.4), (1.6) and (1.7) as well as the condition (2.12b), we have by (2.22) and (2.14)

$$
\begin{aligned}
(2.23)\ & B(q_0;\tau_{z_1},\cdots,\tau_{z_{m-k}};\mu) \\
&\le \sum_{j_1+\cdots+j_{m-k+1}=q_0} \int_{\Delta^{z_1,\cdots,z_{m-k}}_{q_0;j_1,\cdots,j_{m-k+1}}} \Big[\prod_{l=1}^{m-k} \|\theta(\tau_{z_l},\cdot)\|_\infty^{q_l-1}\|\theta(\tau_{z_l},\cdot)\|_1\Big] \\
&\qquad \Big[\prod_{i=1}^{q_0}\|\theta(s_i,\cdot)\|_1\Big](\lambda/2\pi)^{(q_0+m-k+1)/2} \\
&\qquad [(s_1-a)\cdots(\tau_{z_1}-s_{j_1})(s_{j_1+1}-\tau_{z_1})\cdots(b-s_{q_0})]^{-\frac{1}{2}} X^{q_0}_{i=1} d|\mu|(s_i) \\
&= (\lambda/2\pi)^{(q_0+m-k+1)/2}\Big[\prod_{l=1}^{m-k}\|\theta(\tau_{z_l},\cdot)\|_\infty^{q_l-1}\|\theta(\tau_{z_l},\cdot)\|_1\Big] \\
&\qquad \Big\{\sum_{j_1+\cdots+j_{m-k+1}=q_0} \int_{\Delta^{z_1,\cdots,z_{m-k}}_{q_0;j_1,\cdots,j_{m-k+1}}} \Big[\prod_{j=1}^{q_0}\phi(s_j)\Big]
\end{aligned}
$$

$$[(s_1 - a) \cdots (\tau_{z_1} - s_{j_1})(s_{j_1+1} - \tau_{z_1}) \cdots (b - s_{q_0})]^{-\frac{1}{2}} X_{i=1}^{q_0} ds_i \Big\}$$

where we recall that $\phi(s) = \|\theta(s,\cdot)\|_1 \frac{d|\mu|}{dm_L}(s)$. If we apply the Hölder inequality and the Schwarz inequality to (2.23), then

$$(2.24) \quad B(q_0; \tau_{z_1}, \cdots, \tau_{z_{m-k}}; \mu)$$

$$\leq (\lambda/2\pi)^{(q_0+m-k+1)/2} \Big[\prod_{l=1}^{m-k} \|\theta(\tau_{z_l},\cdot)\|_\infty^{q_l-1} \|\theta(\tau_{z_l},\cdot)\|_1 \Big]$$

$$\sum_{j_1+\cdots+j_{m-k+1}=q_0} \Big\{ \int_{\Delta^{z_1,\cdots,z_{m-k}}_{q_0;j_1,\cdots,j_{m-k+1}}} \prod_{j=1}^{q_0} [\phi(s_j)]^r X_{i=1}^{q_0} ds_i \Big\}^{1/r}$$

$$\Big\{ \int_{\Delta^{z_1,\cdots,z_{m-k}}_{q_0;j_1,\cdots,j_{m-k+1}}} [(s_1 - a) \cdots (\tau_{z_1} - s_{j_1})(s_{j_1+1} - \tau_{z_1}) \cdots$$

$$(b - s_{q_0})]^{-r'/2} X_{i=1}^{q_0} ds_i \Big\}^{1/r'}$$

$$\leq (\lambda/2\pi)^{(q_0+m-k+1)/2} \Big[\prod_{l=1}^{m-k} \|\theta(\tau_{z_l},\cdot)\|_\infty^{q_l-1} \|\theta(\tau_{z_l},\cdot)\|_1 \Big]$$

$$\Big\{ \sum_{j_1+\cdots+j_{m-k+1}=q_0} \Big[\int_{\Delta^{z_1,\cdots,z_{m-k}}_{q_0;j_1,\cdots,j_{m-k+1}}} \prod_{j=1}^{q_0} [\phi(s_j)]^r X_{i=1}^{q_0} ds_i \Big]^{2/r} \Big\}^{\frac{1}{2}}$$

$$\Big\{ \sum_{j_1+\cdots+j_{m-k+1}=q_0} \Big[\int_{\Delta^{z_1,\cdots,z_{m-k}}_{q_0;j_1,\cdots,j_{m-k+1}}} [(s_1 - a) \cdots$$

$$(\tau_{z_1} - s_{j_1})(s_{j_1+1} - \tau_{z_1}) \cdots (b - s_{q_0})]^{-r'/2} X_{i=1}^{q_0} ds_i \Big]^{2/r'} \Big\}^{\frac{1}{2}}.$$

We note that for $0 < p \leq 2$ and for nonnegative reals $a_1, a_2, \cdots, a_n$,

$$(2.25) \qquad \sum_{j=1}^{n} a_j^p \leq n^{(2-p)/2} \Big[\sum_{j=1}^{n} a_j^2 \Big]^{p/2}$$

which follows from the Hölder inequality for $0 < (p/2) < 1$. We also note that there are $(q_0 + m - k)!/(q_0!(m-k)!)$ terms in the sum

$\sum_{j_1+\cdots+j_{m-k+1}=q_0}$ [5, p.36]. From (2.25), we get

(2.26)

$$\Big\{ \sum_{j_1+\cdots+j_{m-k+1}=q_0} \Big[\int_{\Delta^{z_1,\cdots,z_{m-k}}_{q_0;j_1,\cdots,j_{m-k+1}}} \prod_{j=1}^{q_0} [\phi(s_j)]^r X_{i=1}^{q_0} ds_i \Big]^{2/r} \Big\}^{\frac{1}{2}}$$

$$\le \Big\{ \Big[\frac{(q_0+m-k)!}{q_0!(m-k)!} \Big]^{(2-\frac{2}{r})/2} \Big[\sum_{j_1+\cdots+j_{m-k+1}=q_0}$$

$$\Big[\int_{\Delta^{z_1,\cdots,z_{m-k}}_{q_0;j_1,\cdots,j_{m-k+1}}} \prod_{j=1}^{q_0} (\phi(s_j))^r X_{i=1}^{q_0} ds_i \Big]^2 \Big]^{1/r} \Big\}^{\frac{1}{2}}$$

$$\le \Big[\frac{(q_0+m-k)!}{q_0!(m-k)!} \Big]^{1/2r'} \Big\{ \Big[\sum_{j_1+\cdots+j_{m-k+1}=q_0} \int_{\Delta^{z_1,\cdots,z_{m-k}}_{q_0;j_1,\cdots,j_{m-k+1}}}$$

$$\prod_{j=1}^{q_0} (\phi(s_j))^r X_{i=1}^{q_0} ds_i \Big]^2 \Big\}^{1/2r}$$

$$= \Big[\frac{(q_0+m-k)!}{q_0!(m-k)!} \Big]^{1/2r'} \Big[\frac{1}{q_0!} \int_{[a,b]^{q_0}} \prod_{j=1}^{q_0} (\phi(s_j))^r X_{i=1}^{q_0} ds_i \Big]^{1/r}$$

$$= \Big[\frac{(q_0+m-k)!}{q_0!(m-k)!} \Big]^{1/2r'} (q_0!)^{-1/r} \|\phi\|_r^{q_0}.$$

Furthermore, using (2.25) and the definition of $A(q_0;\tau_1,\cdots,\tau_m;r')$ given in (2.1), we have

(2.27)

$$\Big\{ \sum_{j_1+\cdots+j_{m-k+1}=q_0} \Big[\int_{\Delta^{z_1,\cdots,z_{m-k}}_{q_0;j_1,\cdots,j_{m-k+1}}} [(s_1-a)\cdots(\tau_{z_1}-s_{j_1})$$

$$(s_{j_1+1}-\tau_{z_1})\cdots(b-s_{q_0})]^{-r'/2} X_{i=1}^{q_0} ds_i \Big]^{2/r'} \Big\}^{\frac{1}{2}}$$

$$\le \Big[\frac{(q_0+m-k)!}{q_0!(m-k)!} \Big]^{1/2r} \Big\{ \sum_{j_1+\cdots+j_{m-k+1}=q_0}$$

$$\Big[\int_{\Delta^{z_1,\cdots,z_{m-k}}_{q_0;j_1,\cdots,j_{m-k+1}}} [(s_1-a)\cdots(\tau_{z_1}-s_{j_1})\cdots$$

$$(b - s_{q_0})]^{-r'/2} X_{i=1}^{q_0} ds_i]^2 \Big\}^{1/2r'}$$

$$\leq \left[\frac{(q_0 + m - k)!}{q_0!(m-k)!}\right]^{1/2r} \Big\{ \sum_{j_1+\cdots+j_{m-k+1}=q_0} \int_{\Delta_{q_0; j_1, \cdots, j_{m-k+1}}^{z_1, \cdots, z_{m-k}}}$$

$$[(s_1 - a) \cdots (\tau_{z_1} - s_{j_1}) \cdots (b - s_{q_0})]^{-r'/2} X_{i=1}^{q_0} ds_i \Big\}^{1/r'}$$

$$= \left[\frac{(q_0 + m - k)!}{q_0!(m-k)!}\right]^{1/2r} A(q_0; \tau_{z_1}, \cdots, \tau_{z_{m-k}}; r').$$

Combining (2.21), (2.24), (2.26), and (2.27) we get the norm estimate (2.15) for $\lambda > 0$. This also justifies the use of the Fubini theorem in (2.17).

The rest of the proof follows the proof of Theorem 2.1 of [7] and so will just be outlined. Using the Dominated Convergence Theorem and the estimates (2.15), one easily sees that for $\lambda > 0$, $(I_\lambda(F_n)\Psi)(\xi)$ is a continuous function of ξ and vanishes at ∞. So $I_\lambda(F_n)\Psi$ is a member of $C_0(\mathbb{R})$ and the operator-valued function space integral $I_\lambda(F_n)$ exists as an element of $\mathcal{L}$. In fact, for $\operatorname{Re}\lambda \geq 0$ ($\lambda \neq 0$), $\Psi \in \mathbb{R}$, and $\xi \in \mathbb{R}$, $(K_\lambda(F_n)\Psi)(\xi)$ (defined as the right-hand side of (2.16)) is a continuous function of ξ and, using the Riemann-Lebesgue Theorem as in [7, p.653], vanishes at ∞; that is, $K_\lambda(F_n)\Psi \in C_0(\mathbb{R})$.

Let $\mathcal{M}(\mathbb{R})$ denote the Banach space of $\mathbb{C}$-valued, regular measures defined on the Borel class of $\mathbb{R}$ and equipped with the total variation norm. $\mathcal{M}(\mathbb{R})$ is of course the dual of $C_0(\mathbb{R})$. Fix $\Psi \in L_1(\mathbb{R})$ and let $\mu \in \mathcal{M}(\mathbb{R})$. Set

$$g(\lambda) := \int_{-\infty}^{\infty} K_\lambda(F_n)\Psi)(\xi) d\mu(\xi). \tag{2.28}$$

Then it is easy to show that $g(\lambda)$ is continuous in $\mathbb{C}^+$. Further, using the Fubini Theorem and the Cauchy Integral Theorem, if Δ is any triangular contour in $\mathbb{C}^+$,

$$\int_\Delta g(\lambda) d\lambda = 0. \tag{2.29}$$

Thus, by Morera's Theorem, $g(\lambda)$ is analytic in $\mathbb{C}^+$. Also it is not hard to show that as $\lambda \to -iq$ through $\mathbb{C}^+$, $g(\lambda) \to g(-iq)$, that is,

$K_\lambda(F_n)\Psi \to K_{-iq}(F_n)\Psi$ weakly. We conclude that $I_\lambda^{an}(F_n)$ exists and is given by (2.13), and also (2.15) holds for all $\lambda \in \mathbb{C}^+$. Furthermore, for real $q \neq 0$ $J_q^{an}(F_n)$ exists and is given by (2.13) with λ replaced by $-iq$, and the bound (2.15) holds for $\|J_q^{an}(F_n)\|$ with $|\lambda|$ replaced by $|q|$. This finishes the proof.

REMARK. For $F \equiv 1$, the existence and representation of the operators $I_\lambda^{an}(F)$ and $J_q^{an}(F)$ for all $\lambda \in \mathbb{C}^+$ and all nonzero real q follow from [7, Proposition 2.1]. Further,

$$(2.30)\quad (I_\lambda^{an}(F)\Psi)(\xi) = \left[\frac{\lambda}{2\pi(b-a)}\right]^{\frac{1}{2}} \int_{-\infty}^{\infty} \Psi(v) \exp\left\{-\frac{\lambda(v-\xi)^2}{2(b-a)}\right\} dv$$

and

$$(2.31)\qquad \|I_\lambda^{an}(F)\| \leq \left[\frac{|\lambda|}{2\pi(b-a)}\right]^{\frac{1}{2}} \text{ and } \|J_q^{an}(F)\| \leq \left[\frac{|q|}{2\pi(b-a)}\right]^{\frac{1}{2}}.$$

In fact, the right hand side of (2.15) reduces to (2.31) when $n = 0$.

We finish this section by dealing with the case $\mu = 0$, that is $\eta = \sum_{j=1}^{m} \omega_j \delta_{\tau_j}$ with $a < \tau_1 < \cdots < \tau_m < b$ and $\omega_j \in \mathbb{C}$. This case is already covered by Theorem 2.1 but it is instructive to examine it somewhat in its own right. Note that when $\mu = 0$, the hypothesis (2.12a) is trivially satisfied since $\phi(s)$ is identically 0.

COROLLARY 2.1. *Let $\mu = 0$. Under the hypotheses of Theorem 2.1, the operators $I_\lambda^{an}(F_n)$ and $J_q^{an}(F_n)$ exist for all $\lambda \in \mathbb{C}^+$ and all real $q \neq 0$, respectively. Further for $\lambda \in \mathbb{C}^+$, $\Psi \in L_1(\mathbb{R})$ and $\xi \in \mathbb{R}$,*

$$(2.32)\qquad \begin{aligned} &(I_\lambda^{an}(F_n)\Psi)(\xi) \\ &= \sum_{k=0}^{m-1} \sum_{1 \leq z_1 < \cdots < z_{m-k} \leq m} \sideset{}{'}\sum_{q_1+\cdots+q_{m-k}=n} \left[\frac{\lambda}{2\pi}\right]^{(m-k+1)/2} \\ &\quad \frac{n!\,\omega_{z_1}^{q_1}\cdots\omega_{z_{m-k}}^{q_{m-k}}}{q_1!\cdots q_{m-k}!}[(\tau_{z_1}-a)\cdots(b-\tau_{z_{m-k}})]^{-\frac{1}{2}} \end{aligned}$$

$$\int_{\mathbb{R}^{m-k+1}} \prod_{j=1}^{m-k} [\theta(\tau_{z_j}, v_j)]^{q_j} \Psi(v_{m-k+1})$$

$$\exp\Big\{ -\frac{\lambda}{2} \sum_{j=1}^{m-k+1} \frac{(v_j - v_{j-1})^2}{(\tau_{z_j} - \tau_{z_{j-1}})} \Big\} X_{j=1}^{m-k+1} dv_j$$

$$= \sum_{k=0}^{m-1} \sum_{1 \le z_1 < \cdots < z_{m-k} \le m} \sum'_{q_1 + \cdots + q_{m-k} = n} \frac{n! \omega_{z_1}^{q_1} \cdots \omega_{z_{m-k}}^{q_{m-k}}}{q_1! \cdots q_{m-k}!}$$

$$[C_{(\tau_{z_1} - a)/\lambda} \circ \theta(\tau_{z_1})^{q_1} \circ \cdots \circ C_{(\tau_{z_{m-k}} - \tau_{z_{m-k+1}})/\lambda}$$

$$\circ\, \theta(\tau_{z_{m-k}})^{q_{m-k}} \circ C_{(b - \tau_{z_{m-k}})/\lambda} \Psi](\xi)$$

where $\tau_{z_0} = a$, $\tau_{z_{m+1}} = b$, $v_0 = \xi$ and the prime's on the sum over $q's$ denote the expression as in Lemma 2.3. For $\lambda \in \mathbb{C}^+$, we have

$$\text{(2.33)} \qquad \|I_\lambda^{an}(F_n)\|$$

$$\le \sum_{k=0}^{m-1} \sum_{1 \le z_1 < \cdots < z_{m-k} \le m} \sum'_{q_1 + \cdots + q_{m-k} = n} \Big[\frac{|\lambda|}{2\pi}\Big]^{(m-k+1)/2}$$

$$\frac{n! |\omega_{z_1}|^{q_1} \cdots |\omega_{z_{m-k}}|^{q_{m-k}}}{q_1! \cdots q_{m-k}!} [(\tau_{z_1} - a) \cdots (b - \tau_{z_{m-k}})]^{-\frac{1}{2}}$$

$$\prod_{j=1}^{m-k} \|\theta(\tau_{z_j}, \cdot)\|_\infty^{q_j - 1} \|\theta(\tau_{z_j}, \cdot)\|_1].$$

For real $q \neq 0$, $(J_q^{an}(F_n)\Psi)(\xi)$ is given by the right hand side of (2.32) with $\lambda = -iq$ and the bound (2.33) also holds for $J_q^{an}(F_n)$ with $|\lambda|$ replaced by $|q|$.

Proof. We limit ourself to the computations leading to (2.32) and (2.33) for $\lambda > 0$. Let $\Psi \in L_1(\mathbb{R})$, $\xi \in \mathbb{R}$, and $\lambda > 0$ be given. Now, in view of (1.1), we have

(2.34)

$$(I_\lambda(F_n)\Psi)(\xi)$$

$$= \int_{C_0[a,b]} \Big[\int_{(a,b)} \theta(s, \lambda^{-\frac{1}{2}} x(s) + \xi) d\eta(s) \Big]^n \Psi(\lambda^{-\frac{1}{2}} x(b) + \xi) dm_w(x)$$

$$\overset{\text{(I)}}{=} \int_{C_0[a,b]} \Big[\sum_{j=1}^{m} \omega_j \theta(\tau_j, \lambda^{-\frac{1}{2}} x(\tau_j) + \xi) \Big]^n \Psi(\lambda^{-\frac{1}{2}} x(b) + \xi) dm_w(x)$$

$$\overset{\text{(II)}}{=} \int_{C_0[a,b]} \sum_{k=0}^{m-1} \sum_{1 \le z_1 < \cdots < z_{m-k} \le m} \sum_{q_1 + \cdots + q_{m-k} = n}' \frac{n!}{q_1! \cdots q_{m-k}!}$$

$$\prod_{j=1}^{m-k} [\omega_{z_j} \theta(\tau_{z_j}, \lambda^{-\frac{1}{2}} x(\tau_{z_j}) + \xi)]^{q_j} \Psi(\lambda^{-\frac{1}{2}} x(b) + \xi) dm_w(x)$$

$$\overset{\text{(III)}}{=} \sum_{k=0}^{m-1} \sum_{1 \le z_1 < \cdots < z_{m-k} \le m} \sum_{q_1 + \cdots + q_{m-k} = n}' \frac{n!}{q_1! \cdots q_{m-k}!}$$

$$\Big\{ \int_{C_0[a,b]} \prod_{j=1}^{m-k} [\omega_{z_j} \theta(\tau_{z_j}, \lambda^{-\frac{1}{2}} x(\tau_{z_j}) + \xi)]^{n} \Psi(\lambda^{-\frac{1}{2}} x(b) + \xi) dm_w(x) \Big\}.$$

Step (I) results from writing η as $\sum_{j=1}^{m} \omega_j \delta_{\tau_j}$ and carrying out the integral with respect to $\sum_{j=1}^{m} \omega_j \delta_{\tau_j}$. In (II), we use Lemma 2.3, and the last step (III) in (2.34) comes from the linearity of the Wiener integral. By application of the Wiener integration formula to the right hand side of (III) in (2.34) and by a simple change of variables, we obtain the formula (2.32) for $\lambda > 0$.

From (2.32) for $\lambda > 0$, we have by the condition (2.12b)

$$\begin{aligned} (2.35) \qquad & \|I_\lambda(F_n)\| \\ & \le \sum_{k=0}^{m-1} \sum_{1 \le z_1 < \cdots < z_{m-k} \le m} \sum_{q_1 + \cdots + q_{m-k} = n}' \Big[\frac{\lambda}{2\pi} \Big]^{(m-k+1)/2} \\ & \frac{n! |\omega_{z_1}|^{q_1} \cdots |\omega_{z_{m-k}}|^{q_{m-k}}}{q_1! \cdots q_{m-k}!} [(\tau_{z_1} - a) \cdots (b - \tau_{z_{m-k}})]^{-\frac{1}{2}} \end{aligned}$$

$$\prod_{j=1}^{m-k} [\|\theta(\tau_{z_j},\cdot)\|_\infty^{q_j-1}\|\theta(\tau_{z_j},\cdot)\|_1].$$

REMARK. The last expression in (2.32) is a "disentangling" (see [6]) of the operator $I_\lambda^{a_n}(F_n)$.

3. The main existence theorem

As before, let $\eta \in \mathcal{M}(a,b)$ be such that $\eta = \mu + \sum_{j=1}^{m} \omega_j \delta_{\tau_j}$ where μ is absolutely continuous with respect to Lebesgue measure and $a < \tau_1 < \cdots < \tau_m < b$. Also suppose that $\theta : (a,b) \times \mathbb{R} \to \mathbb{C}$ satisfies conditions (2.12a) and (2.12b). Further, for each nonnegative integer n and $y \in C[a,b]$, let $F_n(y)$ be given by (2.11).

Let $\lambda_0 \in (0,\infty]$ and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an analytic function satisfying

$$\sum_{n=0}^{\infty} |a_n| b_n(|\lambda|) < \infty \tag{3.1}$$

for every $\lambda \in \mathbb{C}_{\lambda_0}^+$ where $b_n(|\lambda|)$ is defined in (2.15). Consider the functional

$$F(y) := f\Big[\int_{(a,b)} \theta(s, y(s))\, d\eta(s)\Big] \tag{3.2}$$

for y in $C[a,b]$; that is

$$F(y) = \sum_{n=0}^{\infty} a_n F_n(y). \tag{3.3}$$

THEOREM 3.1. *Let F be given by (3.3) with the F_n's given by (2.11) and with the assumptions discussed above, particularly (3.1), satisfied. Then for every $\lambda \in (0,\lambda_0)$ and every $\xi \in \mathbb{R}$, $\sum_{n=0}^{\infty} a_n F_n(\lambda^{-\frac{1}{2}} x + \xi)$*

converges absolutely for a.e. $x \in C_0[a,b]$. Also the operators $I_\lambda^{an}(F)$ and $J_q^{an}(F)$ exist, respectively, for all $\lambda \in \mathbb{C}_{\lambda_0}^+$ and all nonzero real q such that $|q| < \lambda_0$. Further, for $\lambda \in \mathbb{C}_{\lambda_0}^+$,

$$I_\lambda^{an}(F) = \sum_{n=0}^{\infty} a_n I_\lambda^{an}(F_n) \tag{3.4}$$

and

$$J_q^{an}(F) = \sum_{n=0}^{\infty} a_n J_q^{an}(F_n) \tag{3.5}$$

with $I_\lambda^{an}(F_n)$, $J_q^{an}(F_n)$ given by (2.13) and where the series in (3.4) and (3.5) satisfy

$$\sum_{n=0}^{\infty} \|a_n I_\lambda^{an}(F_n)\| \le \sum_{n=0}^{\infty} |a_n| b_n(|\lambda|) \tag{3.6}$$

and

$$\sum_{n=0}^{\infty} \|a_n J_q^{an}(F_n)\| \le \sum_{n=0}^{\infty} |a_n| b_n(|q|), \tag{3.7}$$

and so converge in operator norm.

Proof. The proof of this theorem is much like the proof of Theorem 3.1 of [7]. Except for some comments in the next paragraph, we will focus attention on the part of the proof dealing with $\lambda > 0$ and leave it to the reader to consult [7] to see how the rest of the proof proceeds.

Suppose $\lambda \in \mathbb{C}_{\lambda_0}^+$. By (2.15) and (3.1), we have

$$\sum_{n=0}^{\infty} \|a_n I_\lambda^{an}(F_n)\| \le \sum_{n=0}^{\infty} |a_n| b_n(|\lambda|) < \infty. \tag{3.8}$$

Hence the right-hand side of (3.4) defines an element of $\mathcal{L}$ for all $\lambda \in \mathbb{C}_{\lambda_0}^+$. Similarly the series in (3.5) defines an element of $\mathcal{L}$ satisfying (3.7) for $|q| < \lambda_0$. Also, since $b_n(|\lambda|)$ is an increasing function of

$|\lambda|$, the series in (3.4) converges uniformly in any compact subset of $\mathbb{C}^+_{\lambda_0}$. This last fact is helpful both in establishing the analyticity of the right-hand side of (3.4) and in showing that the limit as $\lambda \to -iq$ of the right-hand side of (3.4) equals the right-hand side of (3.5).

Now we claim that for $\lambda > 0$,

$$(I_\lambda(F)\Psi)(\xi) = \sum_{n=0}^{\infty} a_n (I_\lambda(F_n)\Psi)(\xi). \tag{3.9}$$

We give the formal argument and then explain the steps.

(3.10)

$$\begin{aligned}
&(I_\lambda(F)\Psi)(\xi) \\
&= \int_{C_0[a,b]} F(\lambda^{-\frac{1}{2}}x+\xi)\Psi(\lambda^{-\frac{1}{2}}x(b)+\xi)dm_w(x) \\
&= \int_{C_0[a,b]} \Big[\sum_{n=0}^{\infty} a_n F_n(\lambda^{-\frac{1}{2}}x+\xi)\Big]\Psi(\lambda^{-\frac{1}{2}}x(b)+\xi)dm_w(x) \\
&= \sum_{n=0}^{\infty} a_n \int_{C_0[a,b]} F_n(\lambda^{-\frac{1}{2}}x+\xi)\Psi(\lambda^{-\frac{1}{2}}x(b)+\xi)dm_w(x) \\
&= \sum_{n=0}^{\infty} a_n (I_\lambda(F_n)\Psi)(\xi).
\end{aligned}$$

The key to justify (3.10) is to see that

$$\begin{aligned}
&\int_{C_0[a,b]} \Big[\sum_{n=0}^{\infty} |a_n|\,|F_n(\lambda^{-\frac{1}{2}}x+\xi)|\Big]|\Psi(\lambda^{-\frac{1}{2}}x(b)+\xi)|dm_w(x) \\
&= \sum_{n=0}^{\infty} |a_n| \int_{C_0[a,b]} |F_n(\lambda^{-\frac{1}{2}}x+\xi)|\,|\Psi(\lambda^{-\frac{1}{2}}x(b)+\xi)|dm_w(x) \\
&\le \sum_{n=0}^{\infty} |a_n| \int_{C_0[a,b]} \Big\{ \sum_{k=0}^{m} \sum_{1\le z_1<\cdots<z_{m-k}\le m} \sideset{}{'}\sum_{q_0+q_1+\cdots+q_{m-k}=n} \\
&\quad \frac{n!|\omega_{z_1}|^{q_1}\cdots|\omega_{z_{m-k}}|^{q_{m-k}}}{q_0!q_1!\cdots q_{m-k}!}
\end{aligned} \tag{3.11}$$

$$\left[\int_{(a,b)} |\theta(s,\lambda^{-\frac{1}{2}}x(s)+\xi)|d|\mu|(s)\right]^{q_0}$$

$$\prod_{l=1}^{m-k} |\theta(\tau_{z_l},\lambda^{-\frac{1}{2}}x(\tau_{z_l})+\xi)|^{q_l}|\Psi(\lambda^{-\frac{1}{2}}x(b)+\xi)|\Big\}dm_w(x)$$

$$\leq \sum_{n=0}^{\infty} |a_n| b_n(|\lambda|)\|\Psi\|_1 < \infty$$

where the last inequality comes from the same argument that yielded the norm inequality (2.15). By considering a Ψ in $L^1(\mathbb{R})$ which never vanishes, we see from (3.11) that for every $\lambda \in (0,\lambda_0)$ and $\xi \in \mathbb{R}$, the series $\sum_{n=0}^{\infty} a_n F_n(\lambda^{-\frac{1}{2}}x+\xi)$ converges absolutely for a.e. $x \in C_0[a,b]$. This is one of the conclusions of the theorem and it also justifies the second equality in (3.10). The interchange of integral and sum in (3.10) follows from (3.11) and the Fubini-Tonelli Theorem. Formula (3.10) is now justified and (3.9) follows immediately.

We now have the basis for showing that $I_\lambda^{an}(F)$ and $J_q^{an}(F)$ exist and are given by (3.4) and (3.5), respectively. However, as mentioned above, we refer the reader to [7, p.658] for this.

COROLLARY 3.1. *Let the conditions of Theorem 3.1 be satisfied. Suppose, in addition, that* $\mu = 0$ *so that* $\eta = \sum_{j=1}^{m} \omega_j \delta_{\tau_j}$. *(Recall that condition (2.12a) is trivially satisfied in this case.) Then the conclusions of Theorem 3.1 are satisfied and, in this case,* $I_\lambda^{an}(F_n)$ *and* $J_q^{an}(F_n)$ *are given by the simpler formulas of Corollary 2.1.*

COROLLARY 3.2. *Let the conditions of Theorem 3.1 be satisfied. Suppose, in addition, that* $\nu = 0$ *so that* $\eta = \mu$. *(Condition (2.12b) is trivially satisfied in this case since there are no* τ_j*'s.) Then the conclusions of Theorem 3.1 are satisfied and, in this case,* $I_\lambda^{an}(F_n)$ *is given by the simpler formula*

$$(3.12)\quad I_\lambda^{an}(F_n)\Psi = n!\int_{\Delta_n} C_{(s_1-a)/\lambda}\circ\theta(s_1)$$
$$\circ\, C_{(s_2-s_1)/\lambda}\circ\theta(s_2)\circ\cdots\circ\theta(s_n)\circ C_{(b-s_n)/\lambda}\,\Psi\, X_{i=1}^{n} d\mu(s_i).$$

Further $J_q^{an}(F_n)\Psi$ *is given by (3.12) but with* λ *replaced by* $-iq$, $0 < |q| < \lambda_0$.

REMARK 3.1. Corollary 3.2 can be obtained from Theorem 3.1 of [7] by regarding $\frac{d|\mu|}{dm_L}$ as part of an adjusted potential θ_1; i.e., $\theta_1(s,\cdot) = \theta(s,\cdot)\frac{d|\mu|}{dm_L}(s)$. However, the other results of this paper do not follow from theorems in [7].

REMARK 3.2. We mentioned in the introduction that we were not striving for maximum generality in this paper and that our results could probably be extended in various directions. We now indicate some possibilities.

(i) We have assumed throughout that the discrete part ν of the measure η is finitely supported. It is probably possible to allow $\nu = \sum_{j=1}^{\infty}\omega_j\delta_{\tau_j}$ where $\sum_{j=1}^{\infty}|\omega_j| < \infty$. Another limit would need to be introduced, but, since $\sum_{j=1}^{m}\omega_j\delta_{\tau_j} \to \sum_{j=1}^{\infty}\omega_j\delta_{\tau_j}$ in total variation norm, it seems likely that it could be handled. The infinite sum would introduce additional combinatorial complications in Theorem 2.1, a result which is already combinatorially involved. One could no longer assume that the τ's are ordered. In fact, for each m, it would be necessary to consider a permutation σ_m of $\{1,\cdots,m\}$ that time-orders the τ's; that is, such that $a < \tau_{\sigma_m(1)} < \cdots < \tau_{\sigma_m(m)} < b$.

(ii) Our assumption that η_{sc}, the singular, continuous part of η, equals 0 could possibly be eliminated. Assuming $\eta_{sc} = 0$ allowed us to reduce the integral in (2.23) with respect to $X_{i=1}^{q_0} d|\mu|(s_i)$ to an integral with respect to Lebesgue measure and then later make use of the explicit calculation in Lemma 2.1. The assumption that $\eta_{sc} = 0$ could perhaps be replaced by a direct assumption on the size of the integral with respect to $X_{i=1}^{q_0} d|\eta_{sc}|(s_i)$.

(iii) In [6] infinite sums of functionals of the form

$$F(y) = \prod_{u=1}^{L}\int_{(a,b)}\theta_u(s,y(s))d\eta_u(s)$$

were considered and shown to form a Banach algebra under a certain norm. Related functionals could almost certainly be considered in the present setting and might form a Banach space. Further, it might well be possible to multiply certain pairs of such functionals and stay within the space. If so, further connections with Feynman's operational calculus could probably be established.

(iv) The first author showed in [4] that the $\mathcal{L}(L_1(\mathbb{R}), C_0(\mathbb{R}))$ theory as developed in [7] enjoys very pleasant stability properties in the θ's and Ψ's. It seems rather likely that these properties would carry over to the present setting along with, possibly, some stability properties in the η's.

References

1. R. H. Cameron and D. A. Storvick, *An operator valued function space integral applied to integrals of functions of class L_2*, J. Math. Anal. Appl., **42**(1973), 330–372.
2. ———, *An operator valued function space integral applied to integrals of functions of class L_1*, Proc. London J. Math., **27**(1973), 345–360.
3. ———, *An operator valued function space integral applied to multiple integrals of functions of class L_1*, Nagoya Math. J., **51**(1973), 345–360.
4. J. S. Chang, *Stability theorems for the Feynman integral: The $\mathcal{L}(L_1(\mathbb{R}), C_0(\mathbb{R}))$ theory*, Supplemento ai Rendiconti del Circolo Matematico di Palermo, Serie II, numero **17**(1987), 135–151.
5. W. Feller, *An introduction to probability theory and its applications*, Vol.I, 2nd ed., New York, John Wiley, 1957.
6. G. W. Johnson and M. L. Lapidus, *Generalized Dyson series, generalized Feynman diagrams, the Feynman integral and Feynman's operational calculus*, Memoirs of Amer. Math. Soc. **62**(1986), 1–78.
7. G. W. Johnson and D. L. Skoug, *The Cameron-Storvick function space integral: The L_1 theory*, J. Math. Anal. Appl., **50**(1975), 647–667.
8. ———, *The Cameron-Storvick function space integral: An $\mathcal{L}(L_p, L_{p'})$ theory*, Nagoya Math. J., **60**(1976), 93–137.
9. M. Reed and B. Simon, *Methods of modern mathematical physics*, Vol.1, Rev. and enl. ed. New York, Academic Press, 1980.
10. E. T. Whittaker and G. N. Watson, *Modern analysis*, Cambridge University Press, Cambridge, 1952.

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