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FINITE GROUP ACTIONS ON SELF-DUAL 4-MANIFOLDS WITH POSITIVE SCALAR CURVATURE

YONG SEUNG CHO

1. Introduction

Let M be a compact, connected, orientable smooth 4-manifold. Let G be a compact semisimple Lie group with Lie algebra $\mathcal{Y}$ and let $\pi : P \to M$ be a principal G-bundle over M. A connection on P is a $\mathcal{Y}$-valued 1-form on P which has horizontal kernel, namely $\omega(i_* A) = A$ where $i_* : \mathcal{Y} \to TP$ is the natural inclusion into the vertical subspace and ω is the projection to the vertical space, and which is equivariant, $g^*\omega(X) = (Adg^{-1})\omega(X)$ for $X \in \Gamma(TP)$ and $g \in G$. Since the difference $A = \nabla_1 - \nabla_2$ of two connections pulls down to M as a Lie algebra valued one form, the set of connections on P forms an affine space $\Gamma(T^*M \otimes \mathcal{Y}) \equiv \Omega^1(Adp)$. A connection determines a covariant derivative $\nabla : \Omega^0(Adp) \to \Omega^1(Adp)$. We extend it to the covariant exterior derivative $d^\nabla : \Omega^p(Adp) \to \Omega^{p+1}(Adp)$ by composing ∇ with exterior multiplication, and we have L^2-adjoint $d^{\nabla *}$ by composing ∇ with contraction. The curvature F_∇ of the connection ∇ is a zero-order operator given by

$$[F_\nabla, \sigma] = \frac{1}{2}\sum_{i \neq j} e^i \wedge e^j (\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i} - \nabla_{[e_i,e_j]})\sigma,$$

here $\{e^i\}$ is a local conframe of a local frame $\{e_i\}$. On the oriented Riemannian 4-manifold M the Hodge star operator $* : \wedge^p - \wedge^{4-p}$ is defined by $\alpha \wedge *\beta = (\alpha, \beta)dvol$ where $\alpha, \beta \in \wedge^p$ and (α,β) is the inner product on P-forms. On 2-forms $*^2 = 1$ and $*$ is conformally invariant. A connection ∇ on a G-bundle $P \to M$ is self dual (anti-self-dual) if its curvature F_∇ is self-dual (anti-self-dual), i.e., $*F_\nabla = F_\nabla(-F_\nabla)$.

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The complex vector bundles (i.e. $U(n)$-bundles) E on M are classified upto topological isomorphism by their rank and their Chern classes $c_1(E) \in H^2(M:\mathbf{Z})$, $c_2(E) \in H^4(M:\mathbf{Z})$. If a $U(n)$-bundle $E \to M$ is reduced to an $SU(n)$-bundle, then the second Chern class $c_2(E)$ determines the bundle E. In the case of $SO(n)$, its double covering Spin(n) is simply connected for $n > 2$ and an $SO(n)$-bundle E on M can be lifted to an Spin(n)-bundle iff its second Stiefel-Whitney class $\omega_2(E) \in H^2(M:\mathbf{Z}_2)$ is zero. If $n > 2$ and $n \neq 4$, then $SO(n)$ is simple and the $SO(n)$ bundles E on M are classified by $\omega_2(E)$ and the first Pontrjagin class $p_1(E)$. The group $SO(4)$ is not simple since $so(4) = so(3) \oplus so(3)$. The $SO(4)$-bundles $E \to M$ are classified by $\omega_2(E)$, $p_1(E)$ and the fourth Stiefel-Whitney class $\omega_4(E) \in H^4(M:\mathbf{Z}_2)$.

For a Riemannian 4-manifold there is a unique Levi-Civita connection and we will denote the Riemannian curvature of M by $R \in \Gamma(\wedge^2 \otimes so(4)) = \Gamma(\wedge^2 \otimes \wedge^2)$. By the Hodge star operator $*$ we split $\wedge^2 = \wedge^2_+ \oplus \wedge^2_-$. The symmetric tensor R is an element of $\mathrm{Sym}^2(\wedge^2_+ \oplus \wedge^2_-)$. By Singer-Thorpe curvature tensor R breaks into 5 irreducible components $(\mathrm{Sym}^2 \wedge^2_+)^0 \oplus (1) + [\wedge^2_+, \wedge^2_-] \oplus (\mathrm{Sym}^2 \wedge^2_-)^0 + (1)$ where $\mathbf{o}$ denotes the traceless elements in the symmetric product. Under this decomposition $R = (W_+, \frac{c}{12}, 2B, W_-, \frac{c}{12})$ where c is the scalar curvature, B is the traceless Ricci tensor, and $W_\pm$ are the self-dual and anti-self-dual components of the conformally invariant Weyl tensor. The four manifold M is called Einstein if $B \equiv \mathbf{o}$, conformally flat if $W \equiv \mathbf{o}$ and self-dual (anti-self-dual) if $W_- \equiv \mathbf{o}(W_+ \equiv \mathbf{o})$. If M is a spin manifold, then the connection induced on the self-dual spin bundle $V_+ \to M$ by the Levi-Civita connection is self-dual iff M is Einstein. Examples of self-dual spaces are S^4 and $\mathbf{P}_2(\mathbf{C})$ with their usual matrics. They are Einstein and have positive scalar curvature. Hitchin has proved that S^4 and $\mathbf{P}_2(\mathbf{C})$ are the only self-dual Einstein manifolds with positive scalar curvature.

Let P is a principal G-bundle over M, $P \times_G G$ the bundle associated to P with fiber G, the G-action on itself by the adjoint one. The space of sections $\Gamma(P \times_G G)$ is called the gauge group $\mathcal{F}$ of P which forms a group under pointwise multiplication. The gauge group $\mathcal{F}$ has a natural action on the space of connection which comes from G-action on its Lie algebra. Atiyah, Drinfeld, Hitchin and Manin provide a description of the moduli space of gauge equivalence classes of self-dual connections on S^4 (when $c_2(E) = -1$) by corresponding holomorphic vector bundles on $\mathbf{P}_3(\mathbf{C})$ (which is called the Ward correspondence). The Donaldson's celebrated theorem on the intersection forms of smooth compact definite 4-manifolds is proved by the topological properties of the moduli space of $SU(2)$-self-dual connections of $c_2(E) = -1$ bundle on the given manifold.

In this paper we assume that M is a compact, simply connected smooth 4-manifold with positive definite intersection form.

In Section 2, we show that each G-equivariant $U(1)$-bundle η over M has a unique G-invariant self-dual connection upto gauge equivalence where G is a finite group. Suppose that a finite group G acts trivially on $H^2(M : \mathbf{Z})$, then the reducible self-dual connections on the moduli space $\mathcal{M}$ are fixed by the G-action.

In Section 3, if a self-dual reducible connection is H-invariant, then the group H acts on $H^*(M : \mathbf{R})$ trivially here $*$ is 0, 1 or H^2_- and on $H^*(M : \eta)$ by complex multiplication. If M is a compact, self-dual Riemannian 4-manifold with positive scalar curvature, then for all self-dual connection ∇ the second cohomology group of the fundamental elliptic complex $\mathrm{o} \to \Omega^0(Adp) \xrightarrow{\nabla} \Omega^1(Adp) \xrightarrow{d^{\nabla}_-} \Omega^2_-(Adp) \to \mathrm{o}$ vanishes.

In Section 4, as an example, we give an $\mathbf{Z}_p$-action on $\mathbf{P}_2(\mathbf{C})$. We construct an $SU(2)$-bundle $F \to \mathbf{P}_2(\mathbf{C})$ with the second Chern number -1 and with $\mathbf{Z}_p$-action. The moduli space of self-dual connections on F is an open cone on $\mathbf{P}_2(\mathbf{C})$. When we choose a $\mathbf{Z}_p$-invariant metric on $\mathbf{P}_2(\mathbf{C})$ this moduli space is a $\mathbf{Z}_p$-space. The cone point is fixed by the $\mathbf{Z}_p$-action.

2. $U(1)$-bundle on 4-manifolds

From the works of Donaldson and Freedman, the simply connected compact smooth 4-manifold M with positive definite intersection form is homeomorphic to a connected sum of n copies of $\mathbf{P}^2(\mathbf{C})$,

$$M \simeq \mathbf{P}^2(\mathbf{C}) \# \cdots \# \mathbf{P}^2(\mathbf{C}).$$

The second cohomology group $H^2(M : Z)$ is the direct sum of n copies of $\mathbf{Z}$, and the intersection form ω can be diagonalized into $(1) \oplus \cdots \oplus (1)$

over the integers. Let G be a finite group and act on M. By the work of Donaldson, the G-action on M can be reduced to that on $\mathbf{P}^2(\mathbf{C})$. We may consider an induced representation of G on the cohomology group $H^2(M:\mathbf{Z})$ preserving the intersection form ω, namely

$$\rho : G \to \mathrm{Aut}(H^2(M:\mathbf{Z}), \omega).$$

Let H be the kernel of the representation. Then H acts trivially on M upto homotopy.

THEOREM 2.1 (CONNER-RAYMOND). *There exists a complex line bundle η over M such that $c_1(\eta)^2[M] = 1$, and the group action H on M can be lifted to the total space η.*

If we consider the direct sum $E = \eta \oplus \eta^{-1}$, we have an $SU(2)$-bundle $E = \eta \oplus \eta^{-1} \to M$ with H-action. Since $c_2(E) = c_1(\eta)\cdot c_1(\eta^{-1}) = -c_1(\eta)^2$, we have $c_2(E)[M] = -1$. By averaging we may choose an H-invariant metric on M. Let $\mathcal{M}$ be the moduli space of the gauge equivalence classes of self-dual connections of E. Then the moduli space $\mathcal{M}$ has an H-action and has a formal 5-dimension. The formal 5-dimensional topological space $\mathcal{M}$ may have singular points because of the non-zero of the second cohomology group $H^2(M:\mathbf{Z})$ and the transversality of the fundamental elliptic operator (cf. Theorem 3.3).

If M is a compact self-dual Riemannian manifold with positive scalar curvature, then the space $\mathcal{M}$ of irredecible self-dual connections on E is a smooth 5-dimensional manifold with H-action. Since H preserves the reducible connections the moduli space $\mathcal{M}$ is a smooth 5-dimensional manifold with n-singular points $p_1, \cdots, p_n$ each of which corresponds to the bundle splitting $E = \eta \oplus \eta^{-1}$.

THEOREM 2.2. *Each H-equivariant $U(1)$-bundle η over M has a unique H-invariant self-dual connection upto gauge equivalence.*

Proof. Let ∇ be a connection on η. By averaging ∇ over H we have an H-invariant connection $\nabla_1 = \dfrac{1}{|H|}\sum_{h\in H} h\nabla h^{-1}$. Locally $\nabla_1 = d + iA$ where A is a real valued one form. The curvature of ∇_1 is $F_{\nabla_1} = idA$ which is H-invariant. The cohomology class $\frac{i}{2\pi}F_{\nabla_1}$ represents the Euler class $c_1(\eta)$ for the bundle η. By the Hodge Theorem there is the unique harmonic form $h \in \Omega^2(M)$ such that $[h] = \frac{i}{2\pi}F_{\nabla_1} = c_1(\eta)$. So

$\frac{i}{2\pi}F_{\nabla_1} - h = dA$ where $A \in \Omega^1(M)$ is H-invariant by averaging. Let $\nabla_2 = \nabla_1 + i2\pi A$. The connection ∇_2 is an H-invariant connection on η. $F_{\nabla_2} = F_{\nabla_1} + i2\pi dA = -2\pi i h$ is an H-invariant harmonic 2-form. Since H acts on M as isometries, H acts on $\Omega^2_\pm(M)$ which are the ± 1-eigenspaces of the Hodge star operator on M. Since M has a positive definite form, $F_{\nabla_2} \in H^2_+(M : R)$ and ∇_2 is a self-dual H-invariant connection on η. Uniqueness, if ∇' is any other H-invariant connection on η with $F_{\nabla_2} = F_{\nabla'}$, then $\nabla' = \nabla_2 + iA'$ here $dA' = o$ and A' is H-invariant. Since M is simply connected $A' = df$ for some real valued function f on M. Thus we have

$$\exp(-if)\nabla_2 \exp(if) = \nabla_2 + idf = \nabla'$$

we complete the proof.

For any manifold M (we treat only compact simply connected 4-manifolds) each cohomology class of $H^2(M : \mathbf{Z})$ corresponds an equivalence class of complex line bundles over M. The $SU(2)$-bundles over a compact oriented 4-manifold M are classified by the cohomology classes in $H^4(M : \mathbf{Z})$ which are their second Chern classes.

THEOREM 2.3. *Suppose that a finite group H acts trivially on $H^2(M : \mathbf{Z})$. Then the reducible self-dual connections on the moduli space $\mathcal{M}$ are fixed by H-action.*

Proof. By construction bundle $E \to M$ has its Euler class $c_2(E) = -1$. The bundle E has n distinct splittings where $n = \text{rank } H^2(M : \mathbf{Z})$. So there are complex line bundles $\eta_1, \cdots, \eta_n$ such that $E = \eta_1 \oplus \eta_1^{-1} = \cdots = \eta_n \oplus \eta_n^{-1}$. By theorem 2.2, there is a unique H-invariant self-dual connection ∇_i for each $U(1)$-bundle η_i such that $c_1(\eta_i) = \frac{i}{2\pi}F_{\nabla_i}$. Then $\nabla^i = \nabla_i \oplus \nabla_i$ $(i = 1, \cdots, n)$ are H-invariant reducible self-dual $SU(2)$-connections on E. The curvature of ∇^i is $F_{\nabla^i} = F_{\nabla_i} \oplus (-F_{\nabla_i})$ $(i = 1, \cdots, n)$. The curvatures F_{∇^i} $(i = 1, \cdots, n)$ are H-invariant and self-dual. Thus we prove the Theorem.

REMARK [1]. Suppose that the bundle E splits, i.e. $E = \eta \oplus \eta^{-1}$. Let ∇_1 be a self-dual H-invariant connection on $\eta \to M$. The Euler class $c_1(\eta) = \frac{i}{2\pi}F_{\nabla_1} \in H^2(M : \mathbf{Z})$ and the Euler class $c_2(E) = -c_1(\eta)^2$.

By Chern-Weil formula $c_2(E) = \frac{1}{8\pi^2}\mathrm{tr}\, F_\nabla \wedge F_\nabla$, where

$$\begin{aligned}\nabla &= \nabla_1 \oplus \overline{\nabla}_1\\ c_2(E) &= \frac{1}{8\pi^2}\mathrm{tr}\begin{pmatrix} F_{\nabla_1} & 0 \\ 0 & -F_{\nabla_1}\end{pmatrix} \wedge \begin{pmatrix} F_{\nabla_1} & 0 \\ 0 & -F_{\nabla_1}\end{pmatrix}\\ &= \frac{1}{4\pi^2} F_{\nabla_1} \wedge F_{\nabla_1}.\end{aligned}$$

Since F_{∇_1} is self-dual we may write locally

$$\begin{aligned}F_{\nabla_1} &= \sum F_{ij} dx_i \wedge dx_j\\ &= F_{12}(dx_1 \wedge dx_2 + dx_3 \wedge dx_4) + F_{13}(dx_1 \wedge dx_3 + dx_4 \wedge dx_2)\\ &\quad + F_{14}(dx_1 \wedge dx_4 + dx_2 \wedge dx_3)\end{aligned}$$

Thus $c_2(E) = \frac{1}{2\pi^2}(F_{12}^2 + F_{13}^2 + F_{14}^2) dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$, where the F_{ij}'s are imaginary number valued functions on M. Hence $-\frac{1}{2\pi^2}\int_M (F_{12}^2 + F_{13}^2 + F_{14}^2) dvol = 1$.

[2]. Suppose that two connections ∇_1 and ∇_2 are H-invariant and that $g(\nabla_1) = \nabla_2$ where g is a gauge transformation. Then $h[g(\nabla_1)] = h[\nabla_2] = \nabla_2 = g(\nabla_1) = g[h(\nabla_1)]$. Thus $[g,h](\nabla_1) = \nabla_1$ and the gauge transformation $[g,h]$ belongs to the isotropy group of ∇_1. If ∇_1 is irreducible, then $[g,h] = \pm 1$. If ∇_1 is reducible, then $[g,h] \in \Gamma_{\nabla_1}$, where $\Gamma_{\nabla_1} = \{g \in \mathcal{F} \mid g(\nabla_1) = \nabla_1\}$ is the isotropy group of ∇_1.

3. Self-dual 4-manifolds with positive scalar curvature

We would like to investigate the H-action near the H-invariant reducible connections. Suppose that the bundle E is of a form of a parallel splitting $E = \eta \oplus \eta^{-1}$ with a corresponding reducible connection $\nabla = \nabla_1 \oplus \overline{\nabla}_1$, where ∇_1 is a self-dual H-invariant $U(1)$-connection on η. Since H acts on each fiber as a complex linear isometry and is parallel, $E = \eta \oplus \eta^{-1}$ is H-parallel splitting. The corresponding Lie algebra bundle is splitted as $\varepsilon \oplus \eta^2$ where ε is the trivial real bundle and $\eta^2 = \eta \otimes \eta$ is a complex line bundle. The isotropy group $\Gamma_\nabla = \{g \in \mathcal{F} \mid g\nabla g^{-1} = \nabla\}$ is isomorphic to S^1. It is clear that the stabilizer and H commute each other. They act on the Adp by the conjugation,

$$\begin{pmatrix} e^{i\theta} & o \\ o & e^{-i\theta} \end{pmatrix}\begin{pmatrix} it & z \\ -z & -it \end{pmatrix}\begin{pmatrix} e^{-i\theta} & o \\ o & e^{i\theta} \end{pmatrix} = \begin{pmatrix} it & e^{2\theta i}z \\ -e^{-2\theta i}\overline{z} & -it \end{pmatrix}$$

Thus they act trivially on ε and act by rotation by 2θ on η^2. The fundamental elliptic complex

$$o \to \Omega^0(Adp) \underset{\nabla^*}{\overset{\nabla}{\rightleftarrows}} \Omega^1(Adp) \xrightarrow{d_-^\nabla} \Omega^2_-(Adp) \to o$$

is Γ_∇ and H-invariant. Thus the cohomology groups $H^*(M : Adp)$ are Γ_∇ and H-representations. According to the bundle splitting, the cohomology groups are also splitted such as $H^*(M : Adp) = H^*(M : R) \oplus H^*(M : \eta^2)$.

THEOREM 3.1. *If a self-dual reducible connection* ∇ *is* H*-invariant, then*

(1) Γ_∇ *and* H *act trivially on* $H^0(M : \mathbf{R})$, $H^1(M : \mathbf{R})$ *and* $H^2_-(M : \mathbf{R})$.

(2) Γ_∇ *and* H *act on* $H^*(M : \eta^2)$ *by complex multiplication.*

This makes $H^*(M : \eta^2)$ *into complex vector spaces. Suppose that the manifold* M *is compact, simply connected and has positive definite intersection form. Then* $H^0(M : \mathbf{R}) = \mathbf{R}$, $H^1(M : \mathbf{R}) = o$ *and* $H^2_-(M : \mathbf{R}) = o$. *By Atiyah-Singer index theorem the fundamental elliptic complex has formal index 5.*

COROLLARY 3.2. *If* M *is a compact simply connected manifold with positive definite intersection form, then*

(1) $H^0(M : \mathbf{R}) = \mathbf{R}$, *on which* Γ_∇ *and* H *act trivially*

(2) $H^1(M : \eta^2) = \mathbf{C}^{p+3}$ *and* $H^2(M : \eta^2) = \mathbf{C}^p$ *on which* Γ_∇ *and* H *act by complex multiplication.*

Furthermore if M *is a self-dual manifold with positive scalar curvature, then*

(3) $H^1(M : \eta^2) = \mathbf{C}^3$ *and* $H^2(M : \eta^2) = o$ *(cf Theorem 3.3)*

THEOREM 3.3 ([AHS]). *Let M be a compact, self-dual, Riemannian manifold of dimension 4 with positive scalar curvature. Then the second cohomology H^2 of the fundamental elliptic complex*

$$o \to \Omega^0(Adp) \underset{\nabla^*}{\overset{\nabla}{\rightleftarrows}} \Omega^1(Adp) \xrightarrow{d_-^\nabla} \Omega^2_-(Adp) \to o$$

vanishes for all self-dual connections. Moreover the moduli space $\mathcal{M}$ is an H-space.

Proof. Replace the above elliptic complex by a single elliptic operator

$$\nabla^* + d_-^\nabla : \Omega^1(Adp) \to \Omega^0(Adp) \oplus \Omega^2_-(Adp)$$

where ∇^* is the L_2-adjoint of ∇. We may write this by the Dirac operator

$$\not{D}: C^\infty(V_+ \otimes V_- \otimes Adp) \to C^\infty(V_- \otimes V_- \otimes Adp)$$

where $V_\pm$ are $\pm\frac{1}{2}$-spinor bundles and $\not{D}(\sigma) = \sum e^i \cdot \nabla_{e_i}\sigma$ defined by the Riemannian connection and the given connection on Adp, and the Clifford multiplication by $\wedge^1$ on V. We may use the same notation $\not{D}$ for its formal adjoint operator.

$$\begin{aligned}\not{D}^2\sigma &= \sum e^i \cdot e^j \cdot \nabla_{e_i}\nabla_{e_j}\sigma \\ &= -\sum \nabla_{e_i}\nabla_{e_i}\sigma + \frac{1}{2}\sum e^i \cdot e^j \cdot (\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i})\sigma\end{aligned}$$

where $\{e^i\}$ is the local dual basis of the local basis $\{e_i\}$ of the tangent bundle. Since $\nabla^*\nabla = -\sum \nabla_{e_i}\nabla_{e_i}$, and $c^2(R) = \frac{1}{2}\sum e^i e^j(\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i})$ we obtain the Weitzenbock formula of

$$\not{D}^2 : \not{D}^2\sigma = \nabla^*\nabla\sigma + c^2(R)\sigma.$$

Suppose that $\not{D}\sigma = o$, then $\not{D}^2\sigma = o$, and

$$(*) \qquad o = \int_M (\not{D}^2\sigma, \sigma) = \int_M (\nabla\sigma, \nabla\sigma) + \int_M (c^2(R)\sigma, \sigma).$$

Here the map

$$c^2(R) : C^\infty(V_- \otimes V_- \otimes Adp) \xrightarrow{R} C^\infty(V_- \otimes V_- \otimes Adp \otimes \wedge^2)$$
$$\xrightarrow{c^2} C^\infty(V_- \otimes V_- \otimes Adp)$$

is an endomorphism.

The curvature

$$R(V_- \otimes V_- \otimes Adp) = R(V_- \otimes V_-) \otimes I + I \otimes R(Adp).$$

Since the connection ∇ on Adp is self-dual, $R(Adp) \in \wedge^2_+(Adp)$. The Clifford multiplication of $\wedge^2_+$ on V_- is trivial, so $c^2(1 \otimes R(Adp)) = o$. Since $R(V_- \otimes V_-) \in C^\infty(\wedge^2 \otimes \wedge^2)$, only the components in $C^\infty(\wedge^2_- \otimes \wedge^2_-)$ act nontrivially on V_-. Since our manifold M is self-dual, the anti-self-dual part of the Weyl curvature tensor $w \equiv o$. Since M has positive scalar curvature, the scalar $c^2(R) > o$. From $(*)$ if $\mathbf{D}\sigma = o$, then $\sigma = o$.

If M is a self-dual manifold with positive scalar curvature, then the moduli space of gauge equivalence classes of self-dual connections has only singular points $p_1, \cdots, p_n$ which come from the bundle splittings. Moreover if we choose a H-invariant metric on M, then the moduli space $\mathcal{M}$ is a 5-dimensional manifold with H-action except the n-singular points $p_1, \cdots, p_n$.

THEOREM 3.4. *If H acts trivially on the cohomology $H^*(M : \mathbf{Z})$, then H has complex representations horizontally and trivial representation vertically on each cone neighbourhood of the singular points $p_1, \cdots, p_n$.*

Proof. By Corollary 3.2 the moduli space has a neighbourhood $\mathbf{c}^3/s^1$ at each singular point. The H-action on the moduli space $\mathcal{M}$ fixes the singular points.

4. Finite group actions on $P_2(\mathbf{C})$

The $SU(2)$-vector bundles on a compact oriented 4-manifold M come from their classifing bundle, the tautological quaternion line bundle $E \to S^4$. By Hitchin's theorem $\mathbf{P}_2(\mathbf{C})$ and S^4 are the only self-dual

Einstein manifolds with positive scalar curvature. Let $f : \mathbf{P}_2(\mathbf{C}) \to S^4$ be a degree one map. The pull-back bundle $F \equiv f^{-1}(E) \to \mathbf{P}_2(\mathbf{C})$ has the second Chern number $c_2(F) = -1$. We would like to investigate finite group actions on $\mathbf{P}_2(\mathbf{C})$ and on the moduli space of gauge equivalence classes of self-dual connections on $F \to \mathbf{P}_2(\mathbf{C})$. Consider finite group actions on $\mathbf{P}_2(\mathbf{C})$. We introduce a theorem of Bredon concerning cyclic group actions on $\mathbf{P}_2(\mathbf{C})$.

THEOREM 4.1 (BREDON). *Let $\mathbf{Z}_p$ be a cyclic group where p is prime. Let $\mathbf{Z}_p$ act on $\mathbf{P}_2(\mathbf{C})$ such that the induced action on the cohomology groups $H^*(\mathbf{P}_2(\mathbf{C}))$ is trivial. Then the fixed point set of $\mathbf{Z}_p$ is one of the followings;*
(I) *a set consisting of one point and a disjoint 2-sphere S^2,*
(II) *a set consisting of three isolated points.*
In particular, the fixed point set of $\mathbf{Z}_2$ is only of type (I).

EXAMPLE 4.2. Consider the action of $\mathbf{Z}_p$ on $\mathbf{P}_2(\mathbf{C})$ defined by $g[z_0, z_1, z_2] = [\omega z_0, z_1, z_2]$ where $\omega = e^{\frac{2\pi i}{p}}$ and g is a generator of $\mathbf{Z}_p$.

Suppose that $[z_0, z_1, z_2]$ is a fixed point of $\mathbf{Z}_p \cdot g[z_0, z_1, z_2] = [\omega z_0, z_1, z_2] = [\lambda z_0, \lambda z_1, \lambda z_2]$ for some complex number $\lambda \neq \mathrm{o}$. If $z_0 \neq \mathrm{o}$ then $\omega = \lambda \neq -1$ and $[z_0, z_1, z_2] = [1, \mathrm{o}, \mathrm{o}]$, and if $z = \mathrm{o}$, then $\lambda = 1$ and $[z_0, z_1, z_2] = [\mathrm{o}, z_1, z_2]$. This action has one isolated fixed point $p = [1, \mathrm{o}, \mathrm{o}]$ and one fixed complex projective line $\mathbf{P}_1(\mathbf{C})$ in $\mathbf{P}_2(\mathbf{C})$. The normal bundle at p may be identified with the space $\{[1, z_1, z_2] \mid z_1, z_2 \in \mathbf{C}\}$. The action of g on the normal bundle is given by $g[1, z_1, z_2] = [\omega, z_1, z_2] = [1, z_1\omega^{-1}, z_2\omega^{-1}]$. Thus the action g on the normal bundle is a rotation through the angle $\frac{2\pi}{p}(p-1)$. Now consider the fixed complex projective line $\mathbf{P}_1(\mathbf{C}) = S^2$ in $\mathbf{P}_2(\mathbf{C})$. The fiber of the normal bundle at a point $[\mathrm{o}, a, b] \in S^2$ may be identified with the space $\{[z, a, b] \mid z \in \mathbf{C}\}$. The action of g on this fiber is given by $g[z, a, b] = [\omega z, a, b]$ and hence the rotation angle associated to g on S^2 is $\frac{2\pi}{p}$. Since any complex line in $\mathbf{P}_2(\mathbf{C})$ has self intersection at a single point, the Euler number of the normal bundle is one.

For each $g \in \mathbf{Z}_p$ the induced action of g on the cohomology groups $H^*(\mathbf{P}_2(\mathbf{C}) : \mathbf{Z})$ is trivial. By Conner and Raymond's theorem there is a $U(1)$-bundle $\eta \to \mathbf{P}_2(\mathbf{C})$ on $\mathbf{P}_2(\mathbf{C})$ such that the square of the first Chern class $c_1(\eta)^2 = 1$ and $\mathbf{Z}_p$ acts on the bundle $\eta \to \mathbf{P}_2(\mathbf{C})$ as a bundle map. Since $SU(2)$-bundle are classified by their second

Chern class, the $SU(2)$-bundle $\eta \oplus \overline{\eta} \to \mathbf{P}_2(\mathbf{C})$ has the second Chern number $c_2(\eta \oplus \overline{\eta}) = -1$ and $\eta \oplus \overline{\eta}$ is isomorphic to the pull back bundle $F = f^{-1}(E)$ obtained by a degree one map $f : \mathbf{P}_2(\mathbf{C}) \to S^4$.

Taubes' existence theorem says that the principal $SU(2)$-bundle over S^4 has a nonnegative Pontrjagin class and irreducible self-dual connections, and hence any principal $SU(2)$-bundle on a definite 4-manifold M with the identical Pontrajagin class has irredcible self-dual connections. The $SU(2)$-bundle $F \to \mathbf{P}_2(\mathbf{C})$ has irreducible connections and the moduli space of gauge equivalence classes of self-dual connections has a dimension 5. In fact the moduli space is an open cone on $\mathbf{P}_2(\mathbf{C})$. The cone point is corresponded by the $U(1)$-bundle $\eta \to \mathbf{P}_2(\mathbf{C})$. By Theorem 2.3 and Theorem 3.3, we have the following theorem.

THEOREM 4.3. *Under the above assumptions*

(I) $\mathbf{Z}_p$ *acts on the moduli space when we take a* $\mathbf{Z}_p$*-invariant metric on* $\mathbf{P}_2(\mathbf{C})$,

(II) *the cone point of the moduli space is fixed by the* $\mathbf{Z}_p$*-action.*

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Department of Mathematics
Ewha Womans University
Seoul 120–750, Korea