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ON THE BOUNDARY BEHAVIOR AND TAYLOR COEFFICIENTS FOR MIXED NORM SPACES $D^{p,q}$

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1. Introduction

Let $U = \{z : |z| < 1\}$ and $T = [-\pi, \pi]$. For $0 < p < \infty$, and $1 \leq q \leq \infty$, the spaces H^q and $D^{p,q}$ are defined to consist of those f holomorphic in U, respectively for which

$$\|f\|_q := \sup_{0 \leq r < 1} M_q(r, f) < \infty$$

and

$$\int_0^1 (1-r)^{p-p/q} M_q(r, f')^p dr < \infty,$$

where

$$M_q(r, f) = (\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{it})|^q dt)^{1/q},$$
$$M_\infty(r, f) = \sup_{t \in T} |f(re^{it})|.$$

By the theorem of Hardy and Littlewood (Theorem 5.11 in [2]), if $0 < p < q$.

$$D^{p,q} = \{f : \int_0^1 (1-r)^{-p/q} M_q(r, f)^p dr < \infty\}.$$

For $0 < s$, $t \leq \infty$, $l(s,t)$ denotes the space of those sequences $\{a_k\}_{k=0}^{\infty}$ for which

$$\{(\sum_{k \in I_m} |a_k|^s)^{1/s}\}_{m=0}^{\infty} \in l^t \quad (s < \infty)$$

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and

$$\{\sup_{k\in I_m} |a_k|\}_{m=0}^{\infty} \in l^t \quad (s=\infty),$$

where $I_m = \{k : 2^m \le k < 2^{m+1}\}$ $(m = 1, 2, \cdots)$ and $I_0 = \{0\}$ (See [5]).

In [1], P. Ahern and M. Jevtic defined mixed norm spaces $D^{p,q}$ and showed that when $q = 2$ these are exactly the spaces D^p introduced by F. Holland and B. Twomey [4]. They also investigated the dual space and multipliers for $D^{p,q}$.

This note is concerned with the tangential boundary behavior and Taylor coefficient conditions of holomorphic functions in connection with $D^{p,q}$. We list some of the known properties of $D^{p,q}$ in the following. Here and throughout this note $\frac{1}{q} + \frac{1}{q'} = 1$ whenever $1 \le q \le \infty$.

PROPOSITION.

(1) *If* $p < q$, *then* $H^p \subset D^{p,q}$.
(2) *If* $p > q$, *then* $H^p \supset D^{p,q}$.
(3) *If* $q \le 2$, *then* $H^q \supset D^{q,q}$.
(4) *If* $2 \le q < \infty$, *then* $H^q \subset D^{q,q}$.
(5) *If* $p_1 \le p_2$, *then* $D^{p_1,q} \supset D^{p_2,q}$.
(6) *If* $q_1 \le q_2$, *then* $D^{p,q_1} \subset D^{p,q_2}$.
(7) *If* $1 \le q \le 2$ *and if* $f(z) = \sum_{n=1}^{\infty} a_n z^n \in D^{p,q}$, *then*

$$\{n^{1/q-1/p} a_n\}_{n=1}^{\infty} \in l(q', p).$$

Proof. See [1. Theorem 6] for (1), (2), (3) and (4). If $f \in D^{p,q}$, then $M_q(r, f') \le C(1-r)^{1/q-1/p-1}$. Hence the proof of (5) is complete. (6) follows from the fact that

$$M_{q_2}(r, f') \le C(1-r)^{1/q_2-1/q_1} M_{q_1}(r, f').$$

See [6. p.48(4')] for (7).

2. Representation and tangential boundary behavior

For $0 \le r < 1$, and $\beta > 0$, let us define

$$G_r^{\beta}(t) = (1 - re^{it})^{-\beta}, \quad t \in T.$$

THEOREM 1. *Let $0 < p \le 2$ and $1 \le q \le 2 \le s < \infty$. If $f'(z) \in D^{p,q}$, then there is an $F(t) \in L^s(T)$ such that*

$$(2.0)\qquad f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(t)G_r^{\beta}(\theta - t)dt = (F * G_r^{\beta})(\theta), \qquad z = re^{i\theta} \in U,$$

where $\beta = 1/p - 1/s$.

Proof. Let $0 < p \le 2$, $1 \le q \le 2$ and let

$$f'(z) = \sum_{n=1}^{\infty} na_n z^{n-1} \in D^{p,q}.$$

Then $f'(z) \in D^{p,2}$ by (6) of Proposition. So

$$\{n^{3/2-1/p}a_n\}_{n=1}^{\infty} \in l(2,p) \subset l(q',p)$$

by (7) of Proposition. If we fix $s; 2 \le s < \infty$, then it follows from Hölder's inequality that

$$\sum_{k\in I_n} |k^{1-\beta}a_k|^{s'} \le (\sum_{k\in I_n} |k^{\frac{3}{2}-\frac{1}{p}}a_k|^{q'})^{\frac{s'}{q'}}(\sum_{k\in I_n}\frac{1}{k})^{1-\frac{s'}{q'}}.$$

But since $\sum_{k\in I_n}\frac{1}{k}$ is bounded independently on n, we conclude that

$$(2.1)\qquad \{n^{1-\beta}a_n\}_{n=1}^{\infty} \in l(s',p),$$

where $\beta = 1/p - 1/s$. Next, set

$$b_n = \Gamma(\beta)\Gamma(n+1)a_n/\Gamma(n+\beta) \quad n = 1, 2, \dots .$$

Then since $\Gamma(\beta)\Gamma(n+1)/\Gamma(n+\beta) = 0(n^{1-\beta})$ and $p \le 2$, we obtain from (2.1)

$$(2.2)\qquad \{b_n\}_{n=1}^{\infty} \in l(s',p) \subset l(s',2).$$

Finally let

$$F(t)=\sum_{n=1}^{\infty} b_n e^{int}, \quad t\in T. \tag{2.3}$$

Then it follows from (2.2),(2.3), and the Kellog's version of the Hausdorff-Young theorem [5] that

$$F(t)\in L^S(T). \tag{2.4}$$

On the other hand, termwise integration gives that

$$f(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi} F(t)(1-e^{-it}z)^{-\beta}dt, \quad z\in U. \tag{2.5}$$

Here termwise integration is justified because $F(t)\in L^S$ and the series expansion of $(1-e^{-it}z)^{-\beta}$ is uniformly convergent whenever $z\in U$ is fixed.

From (2.4) and (2.5) the proof is complete.

REMARK. If we suppose (2.0) for some $F(t)\in L^S(T)$, then by Young's inequality after differentiation of $f(z)$ we get

$$M_q(r,f'')\le C\|F\|_{L^s}\|G_r^{\beta+2}\|_{L^t}, \tag{2.6}$$

where $1/q=1/s+1/t-1$. Since $\|G_r^{\beta+2}\|_{L^t}=O(1-r)^{-2-\beta+1/t}$ ([2. p.65]), by (2.6) we conclude that (2.0) implies

$$M_q(r,f'')^p=O((1-r)^{-1-p+p/q}). \tag{2.7}$$

If we compare (2.7) with the definition of $f'\in D^{p,q}$ we see that our exponent $\beta=1/p-1/s$ is best possible.

THEOREM 2.

(1) *If $f'(z)\in D^{1,q}$ for some q, then $f(z)\in BMOA$. (See [3] for BMOA).*

(2) *If $f'(z)\in D^{1,q}$ for some $q: q\le 2$, then $f(z)$ is continuous in $\{z:|z|\le 1\}$.*

(3) *If $2/3<p<1$, $q\le 2$ and if $f'(z)\in D^{p,q}$, then the limit of $f(z)$ as $z\to e^{i\theta}$ within Ω_γ exists almost everywhere on T, where $\Omega_\gamma=\{re^{it}:1-r>|\sin\frac{\theta-t}{2}|^\gamma\}$ and $\gamma=\frac{1}{2(1/p-1)}$.*

Proof. (1) Let $f'(z) \in D^{1,\infty}$. Then $M_\infty(r, f') = O(1-r)^{-1}$ from the very definition of $D^{1,\infty}$. Hence

$$\int_0^1 (1-r)M_\infty(r, f')^2 dr < \infty.$$

Thus $(1-|z|)|f'(z)|^2 dxdy$ is a Carleson measure, whence $f \in BMOA$. ([3. P.240]).

(2) Let $f'(z) = \sum_0^\infty na_n z^n \in D^{1,2}$. Then $\{n^{1/2}a_n\} \in l(2,1)$ by (7) of Proposition, so that

$$\{a_n\}_{n=1}^\infty \in l(1,1) \tag{2.8}$$

by Hölder's inequality. From (2.8) and the Weierstrass M-test $g(t) = \sum_0^\infty a_n e^{int}$ converges uniformly and becomes a continuous function on T. On the other hand, by Abel's theorem [9, p.229],

$$\lim_{r\to 1} f(re^{it}) = g(t)$$

for every $t \in T$. Hence $f(z)$ is the Poisson integral of the continuous function $g(t)$. Whence $f(z)$ is continuous on $\{z : |z| \leq 1\}$.

(3) Let $2/3 < p < 1$, $q \leq 2$, and $f'(z) \in D^{p,q}$. Take $\gamma = \frac{1}{2(1/p-1)}$. Then by Theorem 1,

$$f(z) = (F * G_r^\beta)(t), \quad z = re^{it}$$

for some $F \in L^S$, $\beta = 1/p - 1/s$. Now the existence of the Ω_γ-limit follows from [8. Theorem A-(a)].

3. Taylor coefficients

THEOREM 3. *Let* $f(z) = \sum_{n=0}^\infty a_n z^n$ *be analytic in* U. *Suppose that*

$$\sum_0^N n^2|a_n|^2 = O(N^\alpha) \tag{3.0}$$

for some $\alpha \geq 0$. *If* $q \leq 2$, *then* $f(z) \in D^{p,q}$ *for all* p *with* $1/p > 1/q + \alpha/2 - 1$.

Proof. If $q \leq 2$,

$$M_q(r, f') \leq M_2(r, f') = (\sum n^2 |a_n|^2 r^{2n})^{1/2}. \tag{3.1}$$

It follows from summation by parts that

$$\sum_{n=0}^{\infty} n^2 |a_n|^2 r^{2n} = \sum_{n=0}^{\infty} (\sum_{k=0}^{n} k^2 |a_k|^2)(r^{2n} - r^{2n+2}) + \lim_{N \to \infty} (\sum_{k=0}^{N} k^2 |a_k|^2) r^{2N}. \tag{3.2}$$

The last term of (3.2) is 0 by (3.0). Thus from (3.1) and (3.2)

$$M_q(r, f')^p \leq C(\sum_{0}^{\infty} n^\alpha (r^{2n} - r^{2n+2}))^{p/2} \leq C(1-r)^{-\alpha p/2}.$$

Hence

$$\int_0^1 (1-r)^{p-p/q} M_q(r, f')^p \, dr < \infty$$

for all p with $p - p/q - \alpha p/2 > -1$.

COROLLARY. *Let* $f(z) = \sum a_n z^n$ *be analytic in* U. *If (3.0) holds for some* $\alpha; 0 < \alpha < 2$, *then* $f \in H^2$.

REMARK. A routine calculation gives that (3.0) is equivalent to the condition

$$\{n^{1-\alpha/2} a_n\}_{n=1}^{\infty} \in l(2, \infty). \tag{3.3}$$

But by (7) of Proposition $f(z) = \sum a_n z^n \in D^{p,q}$, $1 \leq q \leq 2$, should satisfy

$$\{n^{1/q - 1/p} a_n\}_{n=1}^{\infty} \in l(q', p). \tag{3.4}$$

If we take $a_n = n^{(\alpha-3)/2}$ for $n = 2^m$, $m = 0, 1, 2, \ldots$ and $a_n = 0$ otherwise, then $\{a_n\}$ satisfies (3.3) but not (3.4) when $1/p = 1/q + \alpha/2 - 1$. Therefore the exponent p in the result of Theorem 3 is best possible.

THEOREM 4. *If* $f(z) = \sum_0^\infty a_n z^n$ *and if*

$$\{n^{1/q-1/p} a_n\}_{n=1}^\infty \in l(1,p),$$

then $f \in D^{p,q}$.

Proof.

$$(3.5) \qquad \begin{aligned} &\int_0^1 (1-r)^{p-p/q} M_q(r,f')^p dr \\ &\le \int_0^1 (1-r)^{p-p/q} M_\infty(r,f')^p dr \\ &\le \int_0^1 (1-r)^{p-p/q} (\sum n|a_n| r^n)^p dr. \end{aligned}$$

If we apply [7. Theorem A] the last integral of (3.5) is at most a constant times

$$\sum_0^\infty (\sum_{k \in I_n} k^{1/q-1/p} |a_k|)^p,$$

whence completes the proof.

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