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FINITE GROUP ACTIONS ON SPHERES AND THE GOTTLIEB GROUP

JOHN OPREA

Introduction

The Gottlieb group (or evaluation subgroup) of a space was first defined in [G1] where its salient properties were described, including its intimate connection with the Euler characteristic. Specifically, it was shown that the nontriviality of the Gottlieb group of a finite polyhedron suffices to ensure the vanishing of the Euler characteristic. Because an aspherical space can be shown to have Gottlieb group equal to the center of the fundamental group, it then follows that the nonvanishing of the Euler characteristic of a finite aspherical polyhedron entails the triviality of the center of the associated group.

Since Gottlieb's original work, some sporadic attempts have been made to compute Gottlieb groups of various types of spaces. (The importance of the Gottlieb group in fixed point theory, where it is a special case of the Jiang subgroup, has been one motivating factor. See [P].) One such attempt was made by Gottlieb's student G. Lang [L] who, in particular, showed that the Gottlieb groups of the orbit spaces of the action on S^3 of the finite subgroups (e.g. binary polyhedral) of S^3 by translation are the centers of these groups. This leads naturally to the problem of determining Gottlieb groups of orbit spaces of spheres by general free actions. Because such an orbit space resembles an aspherical space up to its dimension we are led to the statement of

THEOREM A. *If H is a finite group which acts freely on an odd sphere, then the Gottlieb group of the orbit space of the action is equal to the center of H.*

In order to prove Theorem A we will employ a lifting result due to Gottlieb [G2] and generalized in [HV] and [H]. We give a straightforward obstruction-theoretic proof of this result which allows us to

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identify obstructions to lifting elements of the Gottlieb group up the Postnikov tower. For the situation of Theorem A we use a group cohomology argument to show that these obstructions vanish, allowing each element of the center of H to be lifted.

Throughout this paper spaces will be connected and of the homotopy type of CW complexes. Hence weak equivalences are homotopy equivalences and basepoints are nondegenerate. Also, we *implicitly* assume that anytime we refer to a Postnikov tower or n^{th} Postnikov term for a space X, such objects actually exist. That is, X satisfies a condition such as being simple, n-simple or nilpotent which guarantees existence. We shall *explicitly* prove, however, that the orbit space of a free finite group action on an odd sphere allows the formation of a tower after the first stage.

1. Preliminaries on the Gottlieb group

In this section we recollect the definition and basic properties of the Gottlieb group (see [G1]).

DEFINITION. The *Gottlieb group* of a space X, denoted $G(X)$, consists of all $\alpha \in \pi_1(X)$ such that there is *an associated map* $A : S^1 \times X \to X$ and a homotopy commutative diagram,

$$\begin{array}{ccc} S^1 \times X & \xrightarrow{A} & X \\ \uparrow & \nearrow_{\alpha \vee 1_X} & \\ S^1 \vee X & & \end{array}$$

PROPERTIES.

(1) $G(X) = \mathrm{Im}(ev_\# : \pi_1(X^X, 1_X) \to \pi_1(X))$ where $ev : X^X \to X$ is evaluation at the basepoint.
(2) $G(X) =$ The set of covering transformations of the universal cover of X which are equivariantly homotopic to the identity (under the identification of covering transformations with $\pi_1(X)$).
(3) Each element of $G(X)$ acts trivially on all $\pi_i(X)$. In particular, $G(X)$ is contained in the center of $\pi_1(X)$.
(4) If $X = K(\pi, 1)$, then $G(X) = Z\pi$ (the center of π).

(5) If X is an H-space, then $G(X) = \pi_1(X)$.
(6) $G(\mathbf{R}P^{2n+1}) = \mathbf{Z}/2; G(L(p,q)) = \mathbf{Z}/p$, where $L(p,q)$ is a 3-dimensional Lens space of type (p,q).
(7) If X is a finite polyhedron and $\chi(X) \neq 0$, then $G(X) = \{1\}$.

To indicate the role played by $G(X)$ in influencing the structure of a space X, we note the following

THEOREM (SEE [G3] AND [O]). *Let h denote the Hurewicz homomorphism. If $h(G(X))$ contains a free summand of $H_1(X;\mathbf{Z})$ of rank n, then*

$$X \simeq Y \times T^n,$$

where T^n denotes the n-torus.

Finally, we mention the following nontriviality result due to Lang.

THEOREM ([L]). *Let Y be a 1-connected topological group and H a finite subgroup with $H \cap Z(Y) \neq \{1\}$. Then, under the action of H on Y by left translation,*

$$H \cap Z(Y) \subseteq G(Y/H).$$

In particular, note that for $Y = S^3$ and $H=$ any of the binary polyhedral subgroups, the theorem implies $G(S^3/H) = ZH$.

2. Gottlieb's lifting theorem

Let $\alpha \in G(X)$ and denote by $A : S^1 \times X \to X$ an associated map with $A|_{S^1} \simeq \alpha$ and $A|_X \simeq 1_X$. The induced map on cohomology gives,

$$A^*(X) = 1 \otimes x + \lambda \otimes x_A$$

where $x \in H^n(X)$ and λ is a chosen generator of $H^1(S^1)$. Note that $x_A \in H^{n-1}(X)$ and, although x_A depends on λ, we do not denote this.

Recall that a fibration $p : E \to B$ is a *principal* $K(\pi, r)$*-fibration* if it is a pullback of the path fibration $K(\pi,r) \to PK(\pi,r+1) \xrightarrow{\rho} K(\pi,r+1)$ via a map $k : B \to K(\pi, r+1)$. If $\imath \in H^{r+1}(K(\pi,r+1);\pi)$ is the characteristic class, then let $k^*(\imath) = \mu \in H^{r+1}(B;\pi)$ and recall that a map $f : Y \to B$ has a lifting $\overline{f} : Y \to E$ if and only if $f^*(\mu) = 0$.

We now present a fundamental lifting result due to Gottlieb [G2] and generalized in [HV] and [H]. Our proof is a straightforward application of obstruction theory for fibrations. Although we state the theorem for S^1, the same result (and proof) holds for S^n.

THEOREM 1. *Let $p : E \to B$ be a principal $K(\pi, r)$-fibration ($r > 0$) and let $A : S^1 \times B \to B$ be a map with $A|_B = 1_B$. Then, there exists a map $\overline{A} : S^1 \times E \to E$ with $\overline{A}|_E = 1_E$ and a commutative diagram*

$$\begin{array}{ccc} S^1 \times E & \xrightarrow{\overline{A}} & E \\ {\scriptstyle 1_{S^1}\times p}\downarrow & & \downarrow{\scriptstyle p} \\ S^1 \times B & \xrightarrow{A} & B \end{array}$$

if and only if $\mu_A = 0 \in H^r(B; \pi)$.

REMARK. Without loss of generality we can take the diagram to be homotopy commutative. This follows from our conventions about the types of spaces involved and the homotopy lifting property.

Proof. Suppose $\overline{A}$ and the commutative diagram exist. Then

$$\begin{aligned} (1 \times p)^* A^*(\mu) &= \bar{A}^* p^*(\mu) \\ &= \bar{A}^* p^* k^*(\imath) \\ &= 0 \end{aligned}$$

since $p^*k^* = (kp)^* = (\rho\overline{k})^*$ with $\overline{k} : E \to PK(\pi, r+1)$ and $PK(\pi, r+1)$ contractible.

Now,

$$\begin{aligned} 0 &= (1 \times p)^* A^*(\mu) \\ &= (1 \times p)^*(1 \otimes \mu + \lambda \otimes \mu_A) \\ &= 1 \otimes p^*(\mu) + \lambda \otimes p^*(\mu_A). \end{aligned}$$

As above, $p^*(\mu) = 0$, so $0 = p^*(\mu_A)$.

Because $p : E \to B$ is a *principal* $K(\pi, r)$-fibration, it is orientable for any coefficients. Hence, we obtain a Serre sequence,

$$\begin{aligned} \to H^s(B;\pi) \to H^s(E;\pi) \to H^s(K(\pi,r);\pi) \to H^{s+1}(B;\pi) \to \cdots \\ \to H^r(B;\pi) \to H^r(E;\pi) \to H^r(K(\pi,r);\pi). \end{aligned}$$

This implies $p^* : H^i(B;\pi) \to H^i(E;\pi)$ is an isomorphism for $i \le r-1$ and *an injection* for $i = r$. Since $\mu_A \in H^r(B;\pi), p^*(\mu_A) = 0$ implies $\mu_A = 0$.

Conversely, suppose $\mu_A = 0$. We wish to construct a lifting $\overline{A}$ in the following diagram:

$$(*) \qquad \begin{array}{ccccc} S^1 \vee E & & \xrightarrow{\alpha \vee 1_E} & & E \\ \downarrow & & \overline{A} & & \downarrow p, \\ S^1 \times E & \xrightarrow{1 \times p} & S^1 \times B & \xrightarrow{A} & B \end{array} \qquad \alpha = A|_{S^1}$$

Because $p : E \to B$ is a principal $K(\pi, r)$-fibration, we may apply obstruction theory to obtain a single obstruction to the existence of $\overline{A}$ lying in $H^{r+1}(S^1 \times E, S^1 \vee E; \pi)$.

In order to identify this obstruction we use the fact that the diagram

$$\begin{array}{ccc} * & \longrightarrow & S^1 \vee E \\ \downarrow & & \downarrow \\ S^1 \times E & \xrightarrow{1} & S^1 \times E \end{array}$$

induces an *injection* (arising from the cohomology sequence of a pair) for $i > 0$,

$$H^i(S^1 \times E, S^1 \vee E; \pi) \to H^i(S^1 \times E, *; \pi) \cong H^i(S^1 \times E; \pi).$$

By the naturality of obstructions to lifting, our obstruction pulls back to an obstruction in $H^{r+1}(S^1 \times E; \pi)$ which is the obstruction to the existence of a lifting in the diagram

$$\begin{array}{ccccc} & \nearrow & E & \longrightarrow & PK \\ & & \downarrow p & \text{pull} & \downarrow \\ S^1 \times E & \xrightarrow{A(1 \times p)} & B & \xrightarrow{k} & K(\pi, r+1). \end{array}$$

(Strictly speaking, we require a lifting which fixes the basepoint $*$. However using the fact that $*$ is nondegenerate and the homotopy lifting property (as well as taking $B \to K(\pi; r+1)$ to be an inclusion) we reduce to the situation above.)

Now, the obstruction here is clearly

$$\begin{aligned}(1\times p)^*A^*k^*(\imath) &= (1\times p)^*A^*(\mu)\\ &= (1\times p)^*(1\otimes\mu+\lambda\otimes\mu_A)\\ &= 1\otimes p^*(\mu)+\lambda\otimes p^*(\mu_A)\\ &= \lambda\otimes p^*(\mu_A)\end{aligned}$$

since $p^*(\mu) = p^*k^*(\imath) = (kp)^*(\imath)$ and $kp \simeq *$.

By assumption $\mu_A = 0$, so the obstruction vanishes. Now, the injectivity of $H^{r+1}(S^1 \times E, S^1 \vee E; \pi) \to H^{r+1}(S^1 \times E; \pi)$ then implies the vanishing of the obstruction to our original problem. Hence, the desired lifting $\overline{A}$ exists.

3. Consequences

In this section we use the lifting theorem of §2 to relate the Gottlieb Group of a space to that of a stage in its Postnikov tower.

THEOREM 2. *Let $X(n)$ denote the n^{th} term in the Postnikov tower of X. Then $G(X(x+1)) \subseteq G(X(n))$ and $G(X(n+1))$ is identified as the elements of $G(X(n))$ which may be lifted via a diagram*

$$\begin{array}{ccc} S^1\times X(n+1) & \xrightarrow{\overline{A}} & X(n+1)\\ {\scriptstyle 1\times p}\downarrow & & \downarrow{\scriptstyle p}\\ S^1\times X(n) & \xrightarrow{A} & X(n)\end{array}$$

where A is an associated map.

Proof. Let $\overline{A}$ be associated to $\alpha \in G(X(n+1))$. We shall show that $\overline{A}$ is the lift of an α-associated map A. Now, given $\overline{A}$ the naturality of Postnikov systems provides a homotopy commutative diagram:

$$\begin{array}{ccc} S^1\times X(n+1) & \xrightarrow{\overline{A}} & X(n+1)\\ {\scriptstyle 1\times p}\downarrow & & \downarrow{\scriptstyle p}\\ S^1\times X(n) & \xrightarrow{A} & X(n)\end{array}$$

where we have used the fact that $S^1(n) = S^1$. Clearly, the restriction of A to S^1 gives $p_{\#}(\alpha) = \alpha$ under the natural identification of fundamental groups. Also, consider the restriction

$$\begin{array}{ccc} X(n+1) & \xrightarrow{1} & X(n+1) \\ p\downarrow & & \downarrow p \\ X(n) & \xrightarrow{A|} & X(n). \end{array}$$

Taking $X(n+1) \to X(n)$ to be an inclusion, we find the obstructions to the existence of a relative homotopy from $A|$ to the identity lie in

$$H^k(X(n), X(n+1); \pi_k(X(n)) = 0.$$

Hence $A| \simeq 1_{X(n)}$ and A is an associated map to α. Therefore $\alpha \in G(X(n))$.

REMARK. If $A : S^1 \times X \to X$ is an associated map for a space X, then the naturality of Postnikov systems provides compatible associated maps for each term and each fibration in the tower. Therefore if, at any stage, we can show that no associated maps lift, then $G(X) = \{1\}$ by Theorem 2.

For example, let $X = X(2)$ be a two-stage Postnikov system and suppose $\alpha \in Z\pi_1 X$. We then obtain an associated map of spaces (with $\pi_1 X = \pi$)

$$S^1 \times K(\pi, 1) \xrightarrow{A} K(\pi, 1)$$

which is induced by a homomorphism

$$\mathbf{Z} \times \pi \xrightarrow{\psi} \pi$$

given by: $\psi(n, x) = \alpha^n x$. (Note that ψ is a homomorphism precisely because $\alpha \in Z\pi$.) Now, $\alpha \in G(X)$ if and only if there is a homotopy commutative diagram

$$\begin{array}{ccc} S^1 \times X & \xrightarrow{\bar{A}} & X \\ \downarrow & & \downarrow \\ S^1 \times K(\pi, 1) & \xrightarrow{A} & K(\pi, 1) \end{array}$$

where $\overline{A}$ is an associated map.

Theorem 2, together with the Remark, may be used to analyze the structure of the Ganea space. This infinite dimensional space was constructed by Ganea [G] to answer negatively the question posed by Gottlieb [G1] of whether a simple space with nontrivial fundamental group must have nontrivial Gottlieb group. (A finite dimensional example has only recently been constructed in [O].)

EXAMPLE. Construct the Ganea space X as a principal $K(\mathbf{Z}/2,2)$-fibration over $\mathbf{R}P(\infty) = K(\mathbf{Z}/2,1)$ induced by the nontrivial element of $H^3(\mathbf{R}P(\infty);\mathbf{Z}/2) \cong \mathbf{Z}/2$. This element is the cube $\imath^3$ of the polynomial algebra generator $\imath$ in degree 1. Let α be the nontrivial element of $\pi_1 X = \mathbf{Z}/2$.

We have $A : S^1 \times K(\mathbf{Z}/2,1) \to K(\mathbf{Z}/2,1)$ as in the Remark and, by Theorem 1, we know the obstruction to the existence of a lift $\overline{A}$ is $\imath^3_A \in H^2(K(\mathbf{Z}/2,1);\mathbf{Z}/2)$. Now, we can compute $\imath^3_A$ by,

$$A^*(\imath^3) = (A^*(\imath))^3 = (1 \otimes \imath + \lambda \otimes 1)^3 = 1 \otimes \imath^3 + \lambda \otimes \imath^2.$$

Hence, $\imath^3_A = \imath^2 \neq 0$. Therefore a lift of A does not exist and $G(X) = \{1\}$.

We now show that, for finite complexes, there are only a finite number of obstructions to lifting elements of the center of the fundamental group to Gottlieb elements.

THEOREM 3. *Let X be a finite complex of dimension n. If $\alpha \in Z\pi_1 X$ can be lifted to $X(n)$, then $\alpha \in G(X)$.*

Proof. By assumption there is a commutative diagram

$$\begin{array}{ccc} S^1 \times X(n) & \xrightarrow{\overline{A}} & X(n) \\ \downarrow & & \downarrow \\ S^1 \times K(\pi,1) & \xrightarrow{A} & K(\pi,1) \end{array}$$

where $\pi = \pi_1 X, A|_{S^1} = \alpha$ and $\overline{A}|_{X(n)} = 1_{X(n)}$. First, we consider the problem of lifting $\overline{A}$ to an associated map $\tilde{A} : S^1 \times X(n+1) \to X(n+1)$. The obstruction to lifting is $\mu_{\overline{A}} \in H^{n+1}(X(n);\pi_{n+1}X)$. Now, the

Whitehead Theorem shows that the map $X \to X(n)$ induces $H_i(X) \cong H_i(X(n))$ for $i \leq n$ and a surjection $H_{n+1}(X) \to H_{n+1}(X(n))$. However, $H_{n+1}(X) = 0$, so $H_{n+1}(X(n)) = 0$ as well. Also, since X is n-dimensional, $H_n(X) \cong H_n(X(n))$ is free abelian. The universal coefficient theorem now gives,

$$\begin{aligned} H^{n+1}(X(n); \pi_{n+1}X) \cong {} & \mathrm{Hom}(H_{n+1}(X(n)), \pi_{n+1}X) \\ & \oplus \mathrm{Ext}(H_n(X(n)), \pi_{n+1}X) = 0. \end{aligned}$$

Therefore $\tilde{A}$ exists.

Now, for $j > n$, the obstruction to a lift from $S^1 \times X(j) \to X(j)$ to $S^1 \times X(j+1) \to X(j+1)$ lies in $H^{j+1}(X(j); \pi_{j+1}X) = 0$ (since $H_{j+1}X(j) = 0$ and $H_j(X(j)) \cong H_j(X) = 0$).

Therefore, if A can be lifted to $\overline{A}$, then A can be lifted to any stage of the Postnikov tower. Of course, at each stage many liftings might exist, but no matter which is chosen it may be lifted further to make all diagrams commutative. Hence we obtain a map

$$(A_j) : S^1 \times \varprojlim X(j) \to \varprojlim X(j)$$

so that $(A_j)|_{S^1} = \alpha$ (by identifying fundamental groups). Let $\phi : X \to \varprojlim X(j)$ denote the standard weak equivalence and note that there then exists a unique homotopy class $\hat{A} : S^1 \times X \to X$ which makes the following diagram homotopy commutative.

$$\begin{array}{ccc} S^1 \times X & \xrightarrow{\hat{A}} & X \\ {\scriptstyle 1\times\phi}\big\downarrow & & \big\downarrow{\scriptstyle \phi} \\ S^1 \times \varprojlim X(j) & \xrightarrow{(A_j)} & \varprojlim X(j) \end{array}$$

Clearly, $\hat{A}|_{S^1} = \alpha$. To see that $\hat{A}|_X \simeq 1_X$, observe that $\dim(X) = n$ provides a bijection of homotopy classes $[X, X] \cong [X(n), X(n)]$ and $\hat{A}|_X$ corresponds to $A_n|_{X(n)} \simeq 1_{X(n)}$. The bijection then implies $\hat{A}|_X \simeq 1_X$.

Hence, $\hat{A}$ is a map associated to $\alpha \in Z\pi_1 X$ and, therefore, $\alpha \in G(X)$.

We close this section with some examples of calculations of $G(X)$, using Theorem 3, which will lead us to our main result in §4.

EXAMPLE 1. $X = \mathbf{R}P(2n+1)$.

$X(1) = \mathbf{R}P(\infty)$ and $X(2n+1)$ is obtained as a principal $K(\mathbf{Z}, 2n+1)$-fibration induced by the nontrivial element $\beta \in H^{2n+2}(\mathbf{R}P(\infty);\mathbf{Z}) \cong \mathbf{Z}/2$. It is easy to see that $\beta = \sigma^{n+1}$ for $0 \neq \sigma \in H^2(\mathbf{R}P(\infty);\mathbf{Z})$. Then, given $A : S^1 \times K(\mathbf{Z}/2, 1) \to K(\mathbf{Z}/2, 1)$ associated to the nonzero element of $\pi_1 X \cong \mathbf{Z}/2$, the obstruction β_A to a lift $\overline{A} : S^1 \times X(2n+1) \to X(2n+1)$ is identified as follows:

$$A^*(\beta) = A^*(\sigma^{n+1}) = (A^*(\sigma))^{n+1} = (1 \otimes \sigma + \lambda \otimes \sigma_A)^{n+1}.$$

However $\sigma_A \in H^1(\mathbf{R}P(\infty);\mathbf{Z}) = 0$, so $A^*(\beta) = 1 \otimes \sigma^{n+1} = 1 \otimes \beta$. Hence, $\beta_A = 0$ and $\overline{A}$ exists. By Theorem 3, $G(\mathbf{R}P(2n+1)) \cong \mathbf{Z}/2$.

EXAMPLE 2. $X = L(p, 2n+1)$.

Here, $L(p, 2n+1)$ denotes the Lens space obtained as the orbit space of the standard action of $\mathbf{Z}/p$ on $S^{2n+1} \subseteq \mathbf{C}^{n+1}$. The same argument as in Example 1 applied to the generator of the fundamental group of the infinite dimensional Lens space shows that $G(X) \cong \mathbf{Z}/p$.

EXAMPLE 3. $X = S^3/Q(8)$.

The quaternion group of order 8, $Q(8) = \{\pm 1, \pm i, \pm j, \pm k \mid i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j\}$ is contained in S^3 and operates on S^3 freely by left multiplication. Each element of $Q(8)$, considered as a homeomorphism of S^3, is homotopic to the identity, so X is a nilpotent 3-dimensional manifold. Indeed X is n-simple for all $n > 1$, so possesses a Postnikov Tower after stage 1. The first k-invariant, and the only one relavant to the determination of $G(X)$, is an element $\mu \in H^4(Q(8);\mathbf{Z}) \cong \mathbf{Z}/8$ (see [CE]).

Now $ZQ(8) = \{\pm 1\}$, so either $G(X) = \{1\}$ or $G(X) = \{\pm 1\} \cong \mathbf{Z}/2$. We may associate to -1 a map $A : S^1 \times K(Q(8), 1) \to K(Q(8), 1)$ as above and consider whether a lift to $\overline{A} : S^1 \times X(3) \to X(3)$ exists. The obstruction lies in $H^3(Q(8);\mathbf{Z}) \cong \mathrm{Ker}(\mathbf{Z} \xrightarrow{\times 8} \mathbf{Z}) = 0$ (see [CE], p.254). Hence, $\overline{A}$ exists, -1 lifts and $G(X) = \mathbf{Z}/2 = ZQ(8)$.

4. The Gottlieb group of spherical orbit spaces

Throughout this section H will denote a finite group acting freely on an odd sphere S^{2n-1}. Of course, the parity of the sphere is no great restriction since the Lefschetz fixed point theorem implies that the only finite group acting freely on an even sphere is $\mathbf{Z}/2$. The Lefschetz Theorem also shows that the action of H on S^{2n-1} is orientation preserving. Note that the three examples given at the end of §3 fit into this framework. Recall that we are interested in proving

THEOREM A. $G(S^{2n-1}/H) = ZH$.

The plan of proof is as follows: we begin by showing that a Postnikov tower for S^{2n-1}/H may be erected after the first stage, allowing Theorem 3 to be applied. We then use a group cohomology argument to show that the obstruction to lifting to the $(2n-1)^{\text{th}}$ stage vanishes for any element in the center of H, implying the result.

Now, precisely, a "Postnikov tower for the space S^{2n-1}/H after the first stage" means a Moore-Postnikov factorization of the classifying map $S^{2n-1}/H \to K(H,1)$ of the universal cover $S^{2n-1} \to S^{2n-1}/H$. As usual, we take the classifying map to be an inclusion. For the theory of Moore-Postnikov factorizations the reader is referred to Spanier [S, Ch.8, §3]. Suffice it to say that, according to Spanier, a Moore-Postnikov factorization for $S^{2n-1}/H \to K(H,1)$ exists if it can be shown that the pair $(K(H,1), S^{2n-1}/H)$ is *simple*; that is, $\pi_1(S^{2n-1}/H) \cong H$ acts trivially on $\pi_i(K(H,1), S^{2n-1}/H)$ for all i.

LEMMA 4. *$(K(H,1), S^{2n-1}/H)$ is $(2n-1)$-connected and simple.*

Proof. The exact homotopy sequence of the pair (K,S), with $K = K(H,1)$ and $S = S^{2n-1}/H$,

$$\cdots \to \pi_{i+1}(K,S) \to \pi_i(S) \to \pi_i(K) \to \pi_i(K,S) \to \pi_{i-1}(S) \to \cdots$$

shows that,

(1) $\pi_1(K,S) = 0$ since the map $S \to K$ induces an isomorphism of fundamental groups.

(2) $\pi_2(K,S) = 0$ since $\pi_2(K) = 0$ and $\pi_1(S) \to \pi_1(K)$ is an isomorphism.

(3) $\pi_i(K,S) = 0$ for $3 \le i \le 2n-1$ since $\pi_i(K,S) \cong \pi_{i-1}(S) \cong \pi_{i-1}(S^{2n-1}) = 0$ for $i-1 < 2n-1$.

Hence (K,S) is $(2n-1)$-connected. Now, the homotopy sequence is a $\pi_1(S)$-operator sequence, so the isomorphism $\pi_i(K,S) \cong \pi_{i-1}(S)$ (for all $i \ge 3$) is compactible with the respective actions of $\pi_1(S)$. The action of $\pi_1(S)$ on $\pi_{i-1}(S)$, however, is compatible with the covering transformation action of H on $[S^n, S^n]$ (see [S; p.383]) and, since each element of H is orientation preserving, this action is trivial. Hence, the action of $\pi_1(S)$ on $\pi_i(K, S)$ is trivial for all i and (K, S) is simple.

COROLLARY 5. *$S^{2n-1}/H \to K(H,1)$ has a Moore-Postnikov factorization,*

$$
\begin{array}{ccc}
 & & \vdots \\
 & & \downarrow \\
 & \nearrow & S^{2n-1}/H(i+1) \\
 & & \downarrow \\
 & \nearrow & S^{2n-1}/H(i) \\
 & & \vdots \\
 & & \downarrow \\
S^{2n-1}/H & \longrightarrow & K(H,1).
\end{array}
$$

Proof. [S; Thm. 6, p. 444].

The following result will entail the vanishing of our obstructions to lifting Gottlieb elements. Although the result is an immediate consequence of the material in [CE;Ch. 12], we know of no specific reference to a proof, so we provide a proof below.

LEMMA 6. *If H is a finite group which acts freely on a sphere, then $H^{2k-1}(H;\mathbf{Z}) = 0$ for $k \geq 1$.*

Proof. Let $i : S_p \to H$ denote the inclusion of a Sylow p-subgroup and consider the induced map $i^* : H^*(H) \to H^*(S_p)$ and the transfer $\tau : H^*(S_p) \to H^*(H)$ with coefficients understood to be $\mathbf{Z}$. It is well known that the composition τi^* is simply multiplication by the index $[H : S_p]$.

However, by [CE; Thm. 11.6, Ch. 12 and Application 4, p. 357], S_p is either cyclic or generalized quaternion and, therefore (by the explicit calculations [CE; p. 252 and p. 254]), has $H^{2k-1}(S_p) = 0$ for $k \geq 1$.

Thus, $[H : S_p] \cdot H^{2k-1}(H) = 0$ and this is true for al Sylow subgroups. We know each element $\alpha \neq 0$ of $H^{2k-1}(H)$ has order $o(\alpha)$ dividing the order of H, so let p be a prime with $p|o(\alpha)|\ |H|$. Since

$[H : S_p] \cdot H^{2k-1}(H) = 0$, then $p|o(\alpha)|\,[H : S_p]$. This is a contradiction since $[H : S_p]$ is relatively prime to p and so $\alpha = 0$.

Proof of Theorem A. Of course $G(S^{2n-1}/H) \subseteq ZH$, so we must show the reverse inclusion. Let $\alpha \in ZH$ and, as usual, construct the map $A : S^1 \times K(H,1) \to K(H,1)$ corresponding to the homomorphism,

$$\mathbf{Z} \times H \to H, \quad (n, x) \mapsto \alpha^n x.$$

Because $\dim(S^{2n-1}/H) \leq 2n-1$, Theorem 3 assures us that a lifting of A to $\overline{A}$ (below) suffices to show $\alpha \in G(S^{2n-1}/H)$.

$$\begin{array}{ccc} S^1 \times S^{2n-1}/H(2n-1) & \xrightarrow{\overline{A}} & S^{2n-1}/H(2n-1) \\ {\scriptstyle 1\times p}\downarrow & & \downarrow{\scriptstyle p} \\ S^1 \times K(H,1) & \xrightarrow{A} & K(H,1) \end{array}$$

By Corollary 5, p is a principal $K(\mathbf{Z}, 2n-1)$ fibration, so Theorem 1 applies and the obstruction to the existence of $\overline{A}$ lies in $H^{2n-1}(K(H,1); \mathbf{Z}) \cong H^{2n-1}(H; \mathbf{Z})$. By Lemma 6, this group is zero, so the obstruction vanishes and $\overline{A}$ exists. Hence $\alpha \in G(S^{2n-1}/H)$ and $ZH \subseteq G(S^{2n-1}/H)$.

REMARK. Subsequent to the proof of the general result Theorem A, Allen Broughton has given a very nice representation-theoretic proof in the special case of a linear action [B]. For the convenience of the more geometric minded reader, we give Broughton's proof in case H is a subgroup of the unitary group $U(n)$ acting on $\mathbf{C}^n$. Decompose $\mathbf{C}^n$ into irreducibles; $\mathbf{C}^n = \bigoplus_{i=1}^k V_i$. Let $\alpha \in ZH$ and note that, by Schur's Lemma, because α commutes with every element of H, on each V_i α acts as multiplication by a scalar λ_i. Also note that $|\lambda_i| = 1$ since α is of finite order. Now observe that the k-torus $S^1 \times \cdots \times S^1$ acts on $\mathbf{C}^n$ unitarily via each element of the j^{th} circle acting by scalar multiplication on V_j (*and thus commuting with* H). Hence we have,

$$ZH \subseteq S^1 \times \cdots \times S^1 \subseteq U(n).$$

Because ZH is contained in the connected group $S^1 \times \cdots \times S^1$ we can follow the method of Lang [L; Thm.II.4] to show that the covering transformation $\alpha : S^{2n-1} \to S^{2n-1}$ is equivariantly homotopic to the identity. By property (2) of §1, then $\alpha \in G(S^{2n-1}/H)$ and the result is proved. The required homotopy $K : S^{2n-1} \times I \to S^{2n-1}$ is given by

$$K(x,t) = \sigma(t) \cdot x$$

where $\sigma : I \to S^1 \times \cdots \times S^1$ is a path with $\sigma(0) = I$ (the identity) and $\sigma(1) = \alpha$. (Also note that $\sigma(t) \cdot x$ represents the action of the torus on $S^{2n-1} \subseteq \mathbf{C}^n$ as above.) Then $K(x,0) = x$, $K(x,1) = \alpha \cdot x$ and, for $h \in H, hK(x,t) = h\sigma(t) \cdot x = \sigma(t)h \cdot x = \sigma(t) \cdot (h \cdot x) = K(h \cdot x, t)$.

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Cleveland State University
Cleveland, Ohio 44115
U.S.A.