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ON SPECIAL PROJECTIVE KILLING p-FORMS IN RIEMANNIAN MANIFOLDS*

JAE-BOK JUN

1. Introduction

Let M^n $(n \geq 1)$ be an n-dimensional Riemannian manifold and Δ denotes the Laplacian operator. A non-zero p-form u satisfying $\Delta u = \lambda u$ with a constant λ is called a proper form of Δ corresponding to the proper value λ.

In particular, if a function f satisfies $\Delta f = \lambda f$, then it is called the eigenfunction corresponding to the eigenvalue λ. Then, Tachibana has proved the following.

THEOREM A[7]. *In a $2m$-dimensional compact conformally flat Riemannian manifold with positive constant scalar curvature $R = 2m(2m-1)k$, the proper value λ of Δ for m-forms satisfies*

$$\lambda \geq m(m+1)k,$$

and the following relations hold:

$$V^m_{m(m+1)k} = C^m = C^m(d) \oplus K^m, \quad \textit{(direct sum)}.$$

Here and throughout this paper, V^p_λ, C^p etc. denote vector spaces with natural structure defined by

$$\begin{aligned}
V^p_\lambda &= \text{the proper space of } p-\text{forms corresponding to } \lambda,\\
C^p &= \text{the space of all conformal Killing } p-\text{forms},\\
C^p(d) &= \text{the space of all closed conformal Killing } p-\text{forms},\\
K^p &= \text{the space of all Killing } p-\text{forms},\\
K^p_k &= \text{the space of all special Killing } p-\text{forms with } k,\\
SP^p_k &= \text{the space of all special projective Killing } p-\text{forms with } k.
\end{aligned}$$

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If M^n is compact and orientable, the decomposition $V^p_\lambda = (V^p_\lambda \cap d^{-1}(0)) \oplus (V^p_\lambda \cap \delta^{-1}(0))$ holds for $\lambda \neq 0$ from the decomposition theorem of Hodge-de Rham. The purpose of this paper is to introduce that special projective Killing p-forms become proper and find their proper value. In Section 2, we give preliminaries. The Killing, conformal Killing p-forms are recalled in Section 3. We shall discuss the purpose of this paper in Section 4.

2. Preliminaries

Let M^n $(n > 1)$ be an n-dimensional Riemannian manifold. Throughout this paper, manifolds are assumed to be connected and of class C^∞. We denote respectively by g_{bc}, $R_{abc}{}^d$ and $R_{bc} = R_{rbc}{}^r$ the metric, the curvature and the Ricci tensor of a Riemannian manifold. We shall represent tensors by their components with respect to the natural basis, and shall use the summation convention.

For a differential p-form

$$u = \frac{1}{p!} u_{a_1 \cdots a_p} dx^{a_1} \wedge \cdots \wedge dx^{a_p}$$

with skew symmetric coefficients $u_{a_1 \cdots a_p}$, the coefficients of its exterior differential du and the exterior codifferential δu are given by

$$(du)_{a_1 \cdots a_{p+1}} = \sum_{i=1}^{p+1} (-1)^{i+1} \nabla_{a_i} u_{a_1 \cdots \hat{a}_i \cdots a_{p+1}},$$
$$(\delta u)_{a_2 \cdots a_p} = -\nabla^r u_{r a_2 \cdots a_p},$$

where $\nabla^r = g^{rs} \nabla_s$, ∇_s denotes the operator of covariant differentiation, and $\hat{a}_i$ means a_i to be deleted. For p-forms u and v the inner product $\langle u, v \rangle$, the lengths $|u|$ and $|\nabla u|$ are given by

$$\langle u, v \rangle = 1/p! u_{a_1 \cdots a_p} v^{a_1 \cdots a_p}, |u|^2 = \langle u, u \rangle,$$
$$|\nabla u|^2 = \frac{1}{p!} \nabla_b u_{a_1 \cdots a_p} \nabla^b u^{a_1 \cdots a_p}.$$

Denoting by $\Delta = d\delta + \delta d$ the Laplacian operator, we have $\Delta f = -\nabla^r \nabla_r f$ for function f and

$$(\Delta u)_{a_1 \cdots a_p} = -\nabla^r \nabla_r u_{a_1 \cdots a_p} + H(u)_{a_1 \cdots a_p} \tag{2.1}$$

as the coefficients of Δu, where $H(u)_{a_1\cdots a_p}$ are the coefficients of $H(u)$ given by

$$(2.2) \qquad H(u)_a = R_{ar}u^r \quad (p = 1),$$

$$H(u)_{a_1\cdots a_p} = \sum_{i=1}^{p} R_{a_i}{}^r u_{a_1\cdots r\cdots a_p} + \sum_{i<j} R_{a_i a_j}{}^{rs} u_{a_1\cdots r\cdots s\cdots a_p} \quad (n \geq p \geq 2).$$

In the second term on the right-hand side of the last above equation, the subscripts r and s are in the position of a_i and a_j respectively, and we shall use similar arrangements of indices without special notice, (2.1) may be written as follows:

$$(2.3) \qquad \Delta u = -\nabla^r \nabla_r u + H(u).$$

3. The Killing and conformal Killing p-forms

A p-form u ($p \geq 1$) is said to be *Killing* if it satisfies

$$(3.1) \qquad \nabla_b u_{a_1\cdots a_p} + \nabla_{a_1} u_{b a_2\cdots a_p} = 0,$$

which is called the Killing-Yano's equation. Any Killing p-form is co-closed and it is easy to see that (3.1) is equivalent to the following eguation:

$$(3.2) \qquad (du)_{a_1\cdots a_{p+1}} = (p+1)\nabla_{a_1} u_{a_2\cdots a_{p+1}}.$$

It is known that a Killing p-form u satisfies

$$(3.3) \qquad p\nabla^r \nabla_r u + H(u) = 0.$$

Hence, if we take account of (2.3), it follows that

$$(3.4) \qquad p\Delta u = (p+1)H(u).$$

A Killing p-form u ($p \geq 1$) is said to be *special with* k, if it satisfies

$$(3.5) \qquad \nabla_c \nabla_b u_{a_1\cdots a_p} + k\{g_{cb} u_{a_1\cdots a_p} + \sum_{i=1}^{p} (-1)^i g_{ca_i} u_{b a_1\cdots \hat{a}_i \cdots a_p}\} = 0,$$

with a constant k.

For example, any Killing p-form in the sphere of positive constant sectional curvature r is special with $k = r$.

Then it is known that

THEOREM B[8]. *Let M be a complete simply connected Riemannian manifold admitting special Killing p-forms u and v with a positive constant k. If the inner product $\langle u,v\rangle$ is not constant, then M is isometric with $S^n(k)$.*

We shall call a Killing 1-form which is special with constant 1 a Sasakian structure and a Riemannian manifold admitting such a structure is called Sasakian [8]. Moreover, we have proved

LEMMA 3.1[9]. *In any n-dimensional Riemannian manifold, we have*

$$K_k^p \subset V^p_{(p+1)(n-p)k} \quad (n \geq p \geq 1),$$
$$\Delta(d^{-1}(0) \cap \delta^{-1}(K_k^{p-1})) \subset V^p_{p(n-p+1)k} \quad (p > 1),$$

where k is any constant.

A p-form u ($p \geq 1$) is said to be *conformal Killing*, if there exists a $(p-1)$-form θ called the associated form such that

$$\nabla_b u_{a_1\cdots a_p} + \nabla_{a_1} u_{ba_2\cdots a_p} = 2\theta_{a_2\cdots a_p} g_{ba_1} - \sum_{i=2}^{p}(-1)^i(\theta_{ba_2\cdots\hat{a}_i\cdots a_p} g_{a_1a_i} + \theta_{a_1\cdots\hat{a}_i\cdots a_p} g_{ba_i}). \tag{3.6}$$

For a conformal Killing p-form u, the following equations hold

$$\delta u = -(n-p+1)\theta, \tag{3.7}$$

$$(du)_{ba_1\cdots a_p} = (p+1)\{\nabla_b u_{a_1\cdots a_p} + \sum_{i=1}^{p}(-1)^i \theta_{a_1\cdots\hat{a}_i\cdots a_p} g_{ba_i}\}, \tag{3.8}$$

$$p\nabla^r\nabla_r u + H(u) + \frac{2p-n}{n-p+1} d\delta u = 0. \tag{3.9}$$

It should be noticed that (3.8) is equivalent to (3.6).

From (3.6) and (3.7) we have $K^p = C^p \cap \delta^{-1}(0)$.

On the other hand, a simple calculation shows

$$K_k^p \subset d^{-1}(C^{p+1}(d)) \tag{3.10}$$

to be valid for any constant k. Then we have proved the following

LEMMA 3.2[9]. *In any n-dimensional Riemannian manifold, we have*

$$K^p \cap V^p_{(p+1)(n-p)k} \cap d^{-1}(C^{p+1}(d)) = K^p_k \quad (n > p),$$
$$C^p(d) \cap V^p_{p(n-p+1)k} \cap \delta^{-1}(K^{p-1}) \subset \delta^{-1}(K^{p-1}_k) \quad (p > 1),$$

for any constant k.

4. Theorems

An exact p-form $d\theta$ ($p \geq 1$) is said to be ***special projective Killing with constant*** k, if it satisfies the following two equations.

$$(4.1)\quad \begin{aligned} &\nabla_c\nabla_b(d\theta)_{a_1\cdots a_p} \\ &= k\{-g_{cb}(d\theta)_{a_1\cdots a_p} + \sum_{i=1}^{p} g_{ca_i}(d\theta)_{a_1\cdots b\cdots a_p}\} \\ &\quad - (p+1)k\sum_{i=1}^{p}(-1)^{i-1}(g_{ca_i}\nabla_b\theta_{a_1\cdots\hat{a}_i\cdots a_p} + g_{ba_i}\nabla_c\theta_{a_1\cdots\hat{a}_i\cdots a_p}), \end{aligned}$$

$$(4.2)\quad \begin{aligned} &\nabla_c(d\theta)_{ba_2\cdots a_p} + \nabla_b(d\theta)_{ca_2\cdots a_p} \\ &\qquad - (p+1)(\nabla_c\nabla_b\theta_{a_2\cdots a_p} + k\sum_{i=2}^{p} g_{ba_i}\theta_{a_2\cdots c\cdots a_p}) = 0. \end{aligned}$$

Here and in the sequel, let us consider a special projective Killing p-form $d\theta$ ($p \geq 1$) in an n-dimensional Riemannian manifold.

Transvecting (4.1) with g^{cb}, we have,

$$(4.3)\qquad \nabla^r\nabla_r(d\theta)_{a_1\cdots a_p} + k(n+p+2)(d\theta)_{a_1\cdots a_p} = 0.$$

On the other hand, by interchanging indices c and b in (4.1) and making use of the Ricci's identity, we have

$$(4.4)\quad \begin{aligned} &\sum_{i=1}^{p} R_{cba_i}{}^{e}(d\theta)_{a_1\cdots e\cdots a_p} \\ &\qquad + k\{\sum_{i=1}^{p} g_{ca_i}(d\theta)_{a_1\cdots b\cdots a_p} - \sum_{i=1}^{p} g_{ba_i}(d\theta)_{a_1\cdots c\cdots a_p}\} = 0. \end{aligned}$$

Contracting the above equation with g^{ba_1}, we obtain

$$R_c{}^e(d\theta)_{ea_2\cdots a_p} + \frac{1}{2}\sum_{i=2}^{p} R_{ca_i}{}^{rs}(d\theta)_{ra_2\cdots s\cdots a_p} - k(n-p)(d\theta)_{ca_2\cdots a_p} = 0.$$

Also, taking the skew symmetric parts with respect to all the indices in the above equaiton, we can easily verify that

$$\sum_{i=1}^{p} R_{a_i}{}^e(d\theta)_{a_1\cdots e\cdots a_p} + \sum_{i<j} R_{a_ia_j}{}^{rs}(d\theta)_{a_1\cdots r\cdots s\cdots a_p} - p(n-p)k(d\theta)_{a_1\cdots a_p} = 0.$$

Therefore, by virtue of (2.2), the above equation can be rewritten as

$$H(d\theta) - p(n-p)k(d\theta) = 0. \tag{4.5}$$

Hence we have, from (2.3), (4.3) and (4.5)

$$\Delta(d\theta) = (p+1)(n-p+2)k(d\theta),$$

which shows that $d\theta$ is a proper form of Δ corresponding to the proper value $(p+1)(n-p+2)k$. Hence we can conclude the following.

THEOREM 4.1. *In any n-dimensional Riemannian manifold, we have*

$$SP_k^p \subset V^p_{(p+1)(n-p+2)k} \quad (n \geq p \geq 1),$$

where k is any constant.

Next, let w be a closed p-form ($p > 1$) such that $d\delta w$ is special projective Killing with k, that is, $w \in d^{-1}(0)$ and $d\delta w \in SP_k^p$.

Since w is closed, we know $\Delta w \in SP_k^p$. Thus we can obtain from Theorem 4.1 that

$$\Delta\Delta w = (p+1)(n-p+2)k\Delta w,$$

because of $\Delta = d\delta + \delta d$. Hence it holds

THEOREM 4.2. *In any n-dimensional Riemannian manifold, we have*

$$\Delta(SP_k^p) \subset V^p_{(p+1)(n-p+2)} \quad (p > 1),$$

where k is any constant.

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Department of Mathematics Educations
Kookmin University
Seoul 136–702, Korea