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A REMARK ON THE CLIFFORD INDEX AND HIGHER ORDER CLIFFORD INDICES*

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1. Introduction

In [KKM] it was seen that a linear series g^r_{e+2r} computing the Clifford index e of an algebraic curve C is birationally very ample if $r \geq 3$ and $e \geq 3$. The purpose of this present note is to make further observations along the same lines. We also introduce the notion of higher order Clifford indices and make a few remarks on it.

We first fix some basic terminology and notations. C always denote a smooth irreducible projective curve of genus $g \geq 4$. A g^r_d on C is a linear series of degree d and (projective) dimension r on C.
For a line bundle L or a complete g^r_d on C, we define the Clifford index $\mathrm{Cliff}(L)$ of L by

$$\mathrm{Cliff}(L) = \mathrm{Cliff}(g^r_d) = d - 2r = \deg L - 2h^0(C, L) + 2.$$

The Clifford index e of C is defined to be the non-negative integer

$$e := \mathrm{Min}\{\mathrm{Cliff}(L) \mid L \in \mathrm{Pic}(C), h^0(C, L) \geq 2 \text{ and } h^1(C, L) \geq 2\}.$$

We say that a line bundle L (or a complete g^r_d) on C *contributes* to the Clifford index if $h^0(C, L) \geq 2$ and $h^1(C, L) \geq 2$ ($r \geq 1$ and $g-d+r-1 \geq 1$). We say that $L = g^r_d$ on C *computes* the Clifford index of C if L contributes to the Clifford index and $d - 2r = e$; in this case $L = g^r_d$ is obviously base-point-free.

Throughout we work over the field of complex numbers.

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2. Birationally very ample linear series

We start by giving a complement (the case $r = 2$) to [KKM].

PROPOSITION 2.1. *Let a g^2_d on C compute the Clifford index e of C. Then g^2_d is birationally very ample unless C is a 2:1 covering of a smooth plane curve $M \subseteq \mathbf{P}^2$, degree$(M) = e + 4/2$.*

Proof. Assume that $|D| = g^2_d = g^2_{e+4}$ gives a $k : 1$ covering π of a plane curve M, $k \geq 2$. Fix $p \in M$, let t be its multiplicity. Set $E := \pi^*(p)$. Then $|D - E|$ is a base-point-free g^1_{e+4-tk}. Then by definition of the Clifford index, we must have $k = 2$ and $t = 1$.

PROPOSITION 2.2. *Let $|D| = g^r_{2r+e+1}$, $r \geq 3$, be a special linear series without base points on a curve C with Clifford index $e \geq 1$ such that $r(K - D) \geq 1$. Then $|D|$ is birationally very ample.*

Proof. Assume that $|D|$ is not birationally very ample. Then $|D|$ defines a morphism $C \to \mathbf{P}^r$ of degree $m \geq 2$ onto a curve C' in $\mathbf{P}^r$ of degree $d' = \frac{2r+e+1}{n}$. We note that the induced complete linear series $g^r_{d'}$ on C' is very ample since otherwise C' would admit a $g^{r-1}_{d'-2}$ whence there were a $g^{r-1}_{m(d'-2)}$ on C for which $m(d'-2)-2(r-1) = e-2m+3 < e$, a contradiction. Thus C' is a smooth non-degenerate linearly normal curve of degree d' in $\mathbf{P}^r$. In particular $d' \geq r$.

Let $d' \geq r + 2$. Then by the fact that any reduced irreducible and nondegenerate curve of degree at least $r + 2$ in $\mathbf{P}^r$, $r \geq 3$, has an r-secant-$(r-2)$-plane, we get a $g^1_{d'-r}$ on C' inducing a $g^1_{m(d'-r)}$ on C. But $m(d'-r) - 2 = e - 1 - r(m-2) < e$, a contradiction. Thus C' in $\mathbf{P}^r$ is either a rational curve of degree r or an elliptic curve of degree $r + 1$. If C' is rational, C has a g^1_m and we have

$$m - 2 = \frac{e + 2r + 1}{r} - 2 = \frac{e + 1}{r} \geq e$$

which is impossible. If C' is elliptic then C has a g^1_{2m}, and we have

$$2m - 2 = 2\frac{e + 2r + 1}{r + 1} - 2 \geq e$$

which is only possible for $m = 2$ and $e = 1$.

In particular C must be a 2-sheeted cover of an elliptic curve. On the other hand, if $e = 1$ then C is either a smooth plane quintic or a trigonal curve. Since a smooth plane quintic cannot be elliptic-hyperelliptic, C must be trigonal. Because C also has a g_4^1, which is a pull-back of a g_2^1 on C', we have $g(C) \leq (3-1)\cdot(4-1) = 6$ by the Severi's inequality. By Clifford's theorem, we have

$$1 \leq r(K-D) \leq \frac{2g-2-(2r+2)}{2} - 1 = g - r - 3 \leq 3 - r$$

which is contradictory to $r \geq 3$.

We take one step further to prove the following similar result.

PROPOSITION 2.3. *Let $|D| = g^r_{2r+e+2}, r \geq 3$ be a complete linear series without base point on a curve C with Clifford index $e \geq 1$ such that $r(K-D) \geq 1$. Then $|D|$ is birationally very ample unless C is one of the following type:*
(I) C is a triple cover of a curve of genus $g' = 0$ or 1.
(II) C is a double cover of a curve of genus $g' = 2, 3, 4, 5$ or 10.

Proof. Assume that $|D|$ is not birationally very ample. Then $|D|$ defines a morphism $C \to \mathbf{P}^r$ of degree $m \geq 2$ onto a curve C' in $\mathbf{P}^r$ of degree $d' = \frac{2r+e+2}{m}$.

(a) We first consider the case $m \geq 3$:

(i) If $d' \geq r+2$, there is a r-secant-$(r-2)$-plane to C' which induces a $g^1_{d'-r}$ on C' hence a $g^1_{m(d'-r)}$ on C. Then $m(d'-r)-2 = e-(m-2)r < e$, a contradiction.

(ii) In case $m \geq 3$ and $d' = r$, C has a g^1_m and $m - 2 = \frac{e+2}{r} \geq e$, which is only possible for $m = 3$, $r = 3$ and $e = 1$; in other words, C is trigonal.

(iii) In case $m \geq 3$ and $d' = r+1$ with C' an elliptic curve, C has a g^1_{2m} and hence $2m - 2 = 2\frac{e+2r+2}{r+1} - 2 \geq e$, which is only possible for $e = 4$, $m = 3$ and $r = 3$; i.e. C is a triple cover of an elliptic curve.

(b) For the case $m = 2$, C' is birational to a curve C'' in $\mathbf{P}^3$ of degree $\frac{e}{2}+4$ with a complete $g^3_{\frac{e}{2}+4}$ by projecting from $(r-3)$-general points on C' in $\mathbf{P}^r$. We then note the fact that C'' cannot have a

4-secant line; if there were a 4-secant line on C'', there would be a $g^1_{\frac{e}{2}}$ on C'' hence a g^1_e on C, which is a contradiction. By the same computation which was carried out in [M] (lemma 2), we deduce that $(e, g') = (e, g(C'')) = (2,2), (4,4), (2,1), (4,3), (6,5)$ or $(10,10)$. But the case $(e, g') = (2,1)$ does not occur; if this were the case, then g^r_{2r+4} on C is the pull-back of a g^r_{r+2} on C' which is not complete on an elliptic curve C' or C''.

REMARK 2.4. One can easily see that if C is one of the curve described in the statement of the Proposition (2.3), non birationally very ample g^r_{2r+e+2} indeed occur on C just by considering the pull-backs of the appropriate $g^r_{d'}$'s on the base curve C'.

We generalize (2.2) and (2.3) in the following theorem.

THEOREM 2.5. *Let $|D| = g^r_{2r+e+k}$, $r \geq 3$, $k \geq 0$ be a special linear series without base point on a curve C with Clifford index e such that $r(K - D) \geq 1$. If $r \geq 2k + 3$ then $|D|$ is either birationally very ample or 2:1 to a curve of genus $\frac{e+k}{2}$; the last possibility does not occur if $e \geq k + 3$.*

Proof. Assume that $|D|$ is birationally very ample. Then $|D|$ defines a morphism $C \to \mathbf{P}^r$ of degree $m \geq 2$ onto a curve C' in $\mathbf{P}^r$ of degree $d' = \frac{2r+e+k}{m}$ and the induced linear series $g^r_{d'}$ on C' gives rise to a $g^1_{d'-(r-1)}$ by taking off $(r-1)$-general points. Then this in turn induces a $g^1_{2r+e+k-m(r-1)}$ on C whose Clifford index is at most $2r+e+k-m(r-1)-2 = (2-m)(r-1)+e+k$. But if $m \geq 3$, $(2-m)(r-1)+e+k < e$, which is a contradiction.

For the case $m = 2$, we consider the following two cases.

(i) The complete hyperplane series $|D'| = g^r_{r+\frac{e+k}{2}}$ on C' is special: In this case we take off $(r-1)$-general points of C' from the hyperplane series $|D'|$ and then pull it back to C, to get at least a $(r-1)$-dimensional family of g^1_{2+e+k}'s (possibly incomplete) on C. Thus for some $a \geq 1$, there exists at least a $(r-1) - 2(a-1) = (r-1) - \dim \mathbf{G}(1, a)$ dimensional family of complete g^a_{2+e+k}'s on C. In other words, we have

$$\dim W^a_{2+e+k}(C) \geq (r-1) - 2(a-1)$$

and hence

$$\dim W^1_{2+e+k-(a-1)}(C) \geq (r-1) - 2(a-1) + (a-1) = r - a.$$

On the other hand, by applying the basic inequality about the excess linear series (see [FHL]) to the above inequality and by hypothesis $r \geq 2k+3$, $\dim W^1_{e+1}(C) \geq (r-a) - 2(k-a+2) \geq 0$ which is a contradiction.

(ii) If $|D'|$ on C' is non-special, then $g' =$ genus of $C' = \frac{e+k}{2}$. By the existence of a pencil of degree

$$d'' \leq \left[\frac{g'+3}{2}\right] \leq \frac{e+k+6}{4}$$

on C', there exists a pencil of degree at most $\frac{e+k+6}{2}$ on C, hence $\frac{e+k+6}{2} - 2 \geq e$.

3. Higher order Clifford indices

Given a curve C with Clifford index e, set $e_1 := e$. For each $j \in \mathrm{N}$, define $T(j) = \{|D| : 0 < \deg(D) < 2g-2,\ |D|$ and $|K-D|$ are base-point-free and $\mathrm{Cliff}(D) = j\}$. Let $J = \{j \in \mathrm{N} : T(j) \neq \phi\}$. By definition of the Clifford index, $J \neq \phi$ and $e = \min(J)$. For each $j \in J$, we call $|D| \in T(j)$ an admissible linear series. Set $e_2 := \min(J \backslash \{e\})$ if $J \neq \{e\}$. In general if we have defined $e_1, \cdots, e_k$ and $J \neq \{e_1, \ldots, e_k\}$, let $e_{k+1} = \min(J\backslash\{e_1, \ldots, e_k\})$. These integers are called higher order Clifford indices of C, e_k being the k-th Clifford index of C. We call J the Clifford set of C and its cardinality is naturally an invariant of C.

REMARK 3.1. (i) For each $e_j \in J$, let $r_j = \min\{r_j : |D| = g^{r_j}_{d_j} \in T(e_j)\}$. If $g^{r_j}_{d_j}$ is birational for some $j \geq 2$ with $r_j \geq 3$, this condition gives rather strong restrictions on the corresponding map; e.g. every $(2s+2)$-secant s-plane $(0 \leq s \leq r_j - 2)$ must be at leat a $2s+2+(e_j - e_{j-1})$-secant.

(ii) If $e_2 = e+1$, then every $|D| \in T(e_2)$ is birationally very ample by Proposition (2.2).

EXAMPLE 3.2. If C has second Clifford dimension at least 2, i.e. $r_2 > 1$, then an admissible $|D| = g^{r_2}_{2r_2+e_2}$ is birationally very ample unless C is a $e_2 - e + 2$ sheeted cover of a smooth plane curve of degree $\frac{e_2+4}{e_2-e+2}$ and $r_2 = 2$. In particular, if C is a $e_2 - e + 2$ sheeted cover of a smooth conic, then $e_2 = 2e$ and $|D|$ is a double of a pencil g^1_{e+2} which computes the Clifford index.

Proof. Let's assume that the map h induced by $|D|$ has degree $k > 1$. Then $\deg C' = \deg h(C) = \frac{e_2+2r_2}{k}$. Fix a general $p \in C'$ and set $E = h^*(p)$. By construction $|D - E|$ is a base-point-free $g^{r_2-1}_{2r_2+e_2-k}$ and is admissible; we note that no point outside E can be a base point of $|K - D + E|$ since $|K - D|$ has no base point. Then for any $Q \in \mathrm{Supp}(E)$, $h^0(D - E + Q) = h^0(D - E)$ by construction. Furthermore $\mathrm{Cliff}(D - E) = e_2 - k + 2$ and by the minimality of r_2, $k > 2$. Since $\mathrm{Cliff}(D - E) < \mathrm{Cliff}(D)$, we must have $\mathrm{Cliff}(D - E) = e_2 - k + 2 = e$. We now consider the following three cases:

(i) Suppose that $r_2 \geq 4$. Since $|D - E|$ computes the Clifford index, $|D - E|$ is birationally very ample by Theorem 1 of [KKM]. On the other hand $|D - E|$ induces a map of degree at least k by our construction, whence a contradiction.

(ii) If $r_2 = 3$, $|D - E| = g^2_{6+e_2-k} = g^2_{e+4}$. But this cannot happen either since it would induce a $g^1_{e+4-k'}$ $(k' \geq 3)$ on C, which is a pull-back of a pencil $g^1_{\frac{e+4}{k'}-1}$ on the image curve of the map given by $|D - E|$.

(iii) If $r_2 = 2$, C' must be smooth since otherwise C would have $g^1_{e_2+4-tk}$ with $t > 1$. In case $\deg C' = 2$, we have $e_2 = 2e$. Furthermore in this case $|D|$ is a double of a pencil g^1_{e+2} on C which computes the Clifford index.

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