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ON ALMOST EVERYWHERE WARPED PRODUCT MANIFOLDS WITH HARMONIC CURVATURES

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I. Introduction

The notion of warped product manifolds is an important branch of research on differential geometry (see [1] and [2]). The author introduced the notion of almost everywhere warped product manifolds in [4], which is a smooth extension of that of warped product, and studied some fundamental properties of the manifolds.

Recently the research of Riemannian manifolds with harmonic curvatures has become a topic on differential geometry (see [6] and [7]). It is natural for this research to ask that the Ricci tensors of Riemannian manifolds with harmonic curvatures are parallel or not. As an affirmative answer of this question, we shall deal with almost everywhere warped products with harmonic curvatures.

The purpose of this paper is to study a perfect condition for almost everywhere warped product manifolds to have harmonic curvatures or parallel Ricci tensors. Some geometric properties of these manifolds will be investigated, and the so-called Bourguignon's conjecture will be solved negatively by virtue of this study. After recalling the properties of almost everywhere warped products in paragraph II, we shall investigate some conditions to have harmonic curvatures and parallel Ricci tensors in paragraphs III and IV respectively. Paragraph V will be devoted to discuss the Bourguignon's conjecture.

II. Almost everywhere warped products

Let M_1 and $\overline{M}_2$ be Riemannian manifolds of dimensions m and n respectively, and f a positive-valued differential function on M_1 only.

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The *warped product* $M = M_1 \times {}_f\overline{M}_2$ is the product manifold $M_1 \times \overline{M}_2$ endowed with the Riemannian metric

$$(X, X) = (\pi_1 X, \pi_1 X) + f^2(\pi_1 x)(\pi_2 X, \pi_2 X) \tag{2.1}$$

for any vector $X \in T_x(M)$, $x \in M$, where $\pi_a (a = 1, 2)$ are the natural projections $\pi_1 : M \to M_1$, $\pi_2 : M \to \overline{M}_2$, the differential map of π_a is denoted by the same character, and $(\ , \ )$ is the Riemannian inner product. Every surface of revolution (not crossing the axis of revolution) is the typical example of the warped product (see [2]).

Now we shall recall the notion of almost everywhere warped products in [4]. Let M be an $(m+n)$-dimensional Riemannian manifolds, M_1 an m-dimensional submanifold of M, f a differentiable function defined on M_1, N the zero-level hypersurface given by $f = 0$ and M_1^0 a connected component of $M_1 - N$. We assume that the gradient vector field of f does not vanish on N. If $M - N$ is diffeomorphic to the product manifold $M_1^0 \times \overline{M}_2$ of M_1^0 with an n-dimensional Riemannian manifold $\overline{M}_2$, and if the Riemannian metric of M is given by (2.1) on $M - N$, then we say that M is an *almost everywhere warped product* (briefly *AEWP*) of M_1 and $\overline{M}_2$, and denote it by $M = M_1 \times {}_f\overline{M}_2$. We see that AEWP $M = M_1 \times {}_f\overline{M}_2$ is a warped product if the zero-level surface N of f is empty. 2-dimensional Euclidean space R^2 expressed by the usual polar coordinate system is an AEWP $R^2 = R \times {}_f S$ of a real line R and a circle S, where f is the distance function from the origin to any point of R^2. Another examples are given in [4]. Let $(x^A) = (x^h, x^p)$ be a local coordinate system of the AEWP $M = M_1 \times {}_f\overline{M}_2$, called a *separate coordinate*, where (x^h) and (x^p) are those of M_1 and $\overline{M}_2$ respectively. Here and hereafter the indices $A, B, C, D, \cdots ; h, i, j, k, \cdots$ and $p, q, r, s, \cdots$ run over the ranges $1, 2, \cdots, m, m+1, \cdots m+n$; $1, 2, \cdots, m$ and $m+1, m+2, \cdots, m+n$ respectively, unless otherwise stated. If the components of the metric tensors of M, M_1 and $\overline{M}_2$ are denoted by g_{BA}, g_{ji} and $\overline{g}_{qp}$ respectively, then the metric form of the AEWP is expressed by

$$g_{BA} dx^B dx^A = g_{ji} dx^j dx^i + [f(x^h)]^2 \overline{g}_{qp} dx^q dx^p \tag{2.2}$$

with respect to the separate coordinate system. The components of the metric tensor of M belong to (x^p) are equal to

$$g_{qp} = f^2 \overline{g}_{qp}$$

If we denote the Christoffel symbols of M, M_1 and $\overline{M}_2$ by Γ^A_{BC}, $\{{}_j{}^h{}_i\}$ and $\{\overline{{}_r{}^p{}_q}\}$ respectively, then it follows from (2.2) that

$$(2.3)\qquad \begin{cases} \Gamma^h_{ji} = \{{}_j{}^h{}_i\},\ \Gamma^h_{jq} = 0,\ \Gamma^h_{rq} = -ff^h\overline{g}_{rq}, \\ \Gamma^p_{ji} = 0,\ \Gamma^p_{jq} = f^{-1}f_j\delta^p_q,\ \Gamma^p_{rq} = \{\overline{{}_r{}^p{}_q}\}, \end{cases}$$

where we have put

$$f_j = \partial f/\partial x^i \quad \text{and} \quad f^h = g^{ih}f_i.$$

Let D, ∇ and $\overline{\nabla}$ be the Riemannian connections of M, M_1 and $\overline{M}_2$ with respect to the metrics g_{BA}, g_{ji} and $\overline{g}_{qp}$ respectively. The components of curvature tensors of M, M_1 and $\overline{M}_2$ will be denoted by K^A_{DCB}, R^h_{kji} and $\overline{R}^p_{srq}$ respectively. Then, by use of (2.3), we have the relations

$$(2.4)\qquad \begin{cases} K^h_{kji} = R^h_{kji},\ K^h_{sji} = K^p_{krq} = K^h_{sri} = 0, \\ K^h_{krq} = -f(\nabla_k f^h)\overline{g}_{rq},\ K^p_{sji} = -f^{-1}(\nabla_j f_i)\delta^p_s, \\ K^p_{srq} = \overline{R}^p_{srq} - \|G\|^2(\delta^p_s\overline{g}_{rq} - \delta^p_r\overline{g}_{sq}), \end{cases}$$

where $\|\ \|$ indicates the magnitude of a tensor and

$$G = \text{grad } f.$$

It follows from (2.4) that

$$\begin{aligned} \|K^A_{DCB}\|^2 &= \|R^h_{kji}\|^2 + 4nf^{-2}\|\nabla_j G\|^2 \\ &\quad + f^{-4}\|(\overline{R}_{srq}{}^p - \|G\|^2(\delta^p_s\overline{g}_{rq} - \delta^p_r\overline{g}_{sq}))\|^2. \end{aligned}$$

If the function f has non-empty zero-level surface N, then we make a point of M_1 tend to a point on N and obtain the following

THEOREM 2.1([4]). *Let $M = M_1 \times {}_f\overline{M}_2$ be an AEWP of two Riemannian manifolds M_1 and $\overline{M}_2$ of dimensions m and n (≥ 2). If f has non-empty zero-level surface N, then $\overline{M}_2$ is a space of constant curvature, that is,*

$$\overline{R}_{srq}{}^p = \|G\|^2(\delta^p_s\overline{g}_{rq} - \delta^p_r\overline{g}_{sq}).$$

The components of Ricci tensors of M, M_1 and $\overline{M}_2$ will be denoted by K_{CB}, R_{ji} and $\overline{R}_{rq}$ respectively, which are defined by

$$K_{CB} = g^{DA}K_{DCBA},\ R_{ji} = g^{kh}R_{kjih} \text{ and } \overline{R}_{rq} = \overline{g}^{sp}\,\overline{R}_{srqp}.$$

The scalar curvatures K of M, R of M_1 and $\overline{R}$ of $\overline{M}_2$ are defined by

$$K = g^{BA}K_{BA},\ R = g^{ji}R_{ji} \quad \text{and} \quad \overline{R} = \overline{g}^{qp}\,\overline{R}_{qp}.$$

It follows from (2.4) that

$$(2.5)\qquad \begin{cases} K_{ji} &= R_{ji} - nf^{-1}\nabla_j f_i,\ K_{jq} = 0, \\ K_{rq} &= \overline{R}_{rq} - [(n-1)\|G\|^2 + f\Delta f]\overline{g}_{rq}, \end{cases}$$

where Δf is the Laplacian of f. By a simple computation, the covariant derivative of the Ricci tensor of M is given by

(2.6)
$$\begin{cases} D_kK_{ji} &= \nabla_k R_{ji} - nf^{-1}\nabla_k\nabla_j f_i + nf^{-2}f_k\nabla_j f_i, \\ D_sK_{ji} &= D_kK_{jq} = 0, \\ D_sK_{jq} &= -f^{-1}f_j\overline{R}_{sq} + [f_j\Delta f + (n-1)f^{-1}\|G\|^2 f_j - \frac{n}{2}\nabla_j\|G\|^2 \\ &\quad + ff^iR_{ji}]\overline{g}_{rq}, \\ D_kK_{rq} &= -2f^{-1}f_k\overline{R}_{rq} + [f_k\Delta f + 2(n-1)f^{-1}\|G\|^2 f_k \\ &\quad -(n-1)\nabla_k\|G\|^2 - f\nabla_k\Delta f]\overline{g}_{rq}, \\ D_sK_{rq} &= \overline{\nabla}_s\overline{R}_{rq}. \end{cases}$$

III. The harmonic curvatures

For a Riemannian manifold M, if the divergence δK of its curvature tensor K of M vanishes identically, it is said to be *harmonic*. In terms of a local coordinate system (x^A), the divergence δK is expressed by

$$\delta K = D_A K^A_{DCB} = D_D K_{CB} - D_C K_{DB}.$$

Let $M = M_1 \times_f \overline{M}_2$ be an AEWP of two Riemannian manifolds M_1 and $\overline{M}_2$ of dimensions m and n respectively, and assume that M has harmonic curvature. Then it follows from (2.6) that

$$(3.1)\quad \begin{cases} \nabla_k R_{ji} - \nabla_j R_{ki} = -nf^{-1} R^h_{kji} f_h \\ \qquad\qquad -nf^{-2}(f_k \nabla_j f_i - f_j \nabla_k f_i), \\ f_j \overline{R}_{rq} = [(n-1)\|G\|^2 f_j - \frac{1}{2}(n-2) f \nabla_j \|G\|^2 \\ \qquad -f^2 f^i R_{ji} - f^2 \nabla_j \Delta f] \overline{g}_{rq}, \\ \overline{\nabla}_s \overline{R}_{rq} = \overline{\nabla}_r \overline{R}_{sq}. \end{cases}$$

Transvecting f^j to the second relation of (3.1), we have

(3.2)
$$\overline{R}_{rq} = \|G\|^{-2}[(n-1)\|G\|^4 - \frac{1}{2}(n-2) f G \|G\|^2 - f^2 f^j f^i R_{ji} - f^2 G \Delta f] \overline{g}_{rq}.$$

Since $\overline{R}_{rq}$ and $\overline{g}_{rq}$ are quantities of $\overline{M}_2$ and the remanining part of (3.2) is that of M_1, we see that the scalar curvature $\overline{R}$ of $\overline{M}_2$ given by

(3.3)
$$\overline{R} = n\|G\|^{-2}[(n-1)\|G\|^4 - \frac{1}{2}(n-2) f G \|G\|^2 - f^2 f^j f^i R_{ji} - f^2 G \Delta f]$$

is a constant on whole M, and hence $\overline{M}_2$ is Einstein.

From the Ricci identity

$$(3.4)\qquad \nabla_k \nabla_j f_i - \nabla_j \nabla_k f_i = -R^h_{kji} f_h.$$

We have $f^i R_{ji} = -\nabla_j \Delta f + \Delta f_j$, where $\Delta f_h = g^{ji} \nabla_j \nabla_i f_h$. Since it is easily verified that $\Delta \|G\|^2 = 2\|\nabla f\|^2 + 2 f^i \Delta f_i$, we obtain

$$(3.5)\qquad f^j f^i R_{ji} = -G\Delta f + \frac{1}{2}\Delta \|G\|^2 - \|\nabla f\|^2,$$

where $\|\nabla f\|^2 = (\nabla^j f^i)(\nabla_j f_i)$. Substituting (3.5) into (3.3), we have

(3.6)
$$\overline{R} = n\|G\|^{-2}[(n-1)\|G\|^4 - \frac{1}{2}(n-2) f G \|G\|^2 - \frac{1}{2} f^2 \Delta \|G\|^2 + f^2 \|\nabla f\|^2].$$

If the function f has non-empty zero-level surface N, it follows from (3.6) that

$$\overline{R} = n(n-1)\|G\|^2.$$

Summing up the above results, we can state

THEOREM 3.1. *Let $M = M_1 \times_f \overline{M}_2$ be an AEWP of two Riemannian manifolds M_1 and $\overline{M}_2$ of dimensions m and n respectively. Suppose that M has harmonic curvature. Then $\overline{M}_2$ is an Einstein manifold with constant scalar curvature*

$$\overline{R} = n\|G\|^{-2}[(n-1)\|G\|^4 - \frac{1}{2}(n-2)fG\|G\|^2 - \frac{1}{2}f^2\Delta\|G\|^2 + f^2\|\nabla f\|^2]$$

if the zero-level surface N of f is empty, and with constant scalar curvature

$$\overline{R} = n(n-1)\|G\|^2,$$

if the zero-level surface N of f is non-empty.

Now assume that M_1 is an m (≥ 2)-dimensional space of constant curvature R, that is,

$$R^h_{kji} = \frac{R}{m(m-1)}(\delta^h_k g_{ji} - \delta^h_j g_{ki}). \tag{3.7}$$

Then it follows from the first relation of (3.1) and (3.7) that

$$f_k \nabla_j f_i - f_j \nabla_k f_i = -\frac{1}{m(m-1)} Rf(f_k g_{ji} - f_j g_{ki}). \tag{3.8}$$

Applying f^k to (3.8) and summing up with respect to k, we have

$$\|G\|^2 \nabla_j f_i - f_j f^k \nabla_k f_i = -\frac{1}{m(m-1)} Rf(\|G\|^2 g_{ji} - f_j f_i). \tag{3.9}$$

Transvecting g^{ji} to (3.8) again, we obtain

$$\nabla_j \|G\|^2 = 2(\Delta f + \frac{1}{m} Rf) f_j. \tag{3.10}$$

Therefore, comparing (3.9) with (3.10), we have

$$\nabla_j f_i = -\frac{R}{m(m-1)} f g_{ji} + \|G\|^{-2}(\Delta f + \frac{R}{m-1} f) f_j f_i. \tag{3.11}$$

In general a scalar field f satisfying

$$\nabla_j f_i = A g_{ji}$$

is said to be *special concircular*, where A is a scalar field (see [5]).

The equation (3.10) implies that

$$G\|G\|^2 = 2(\Delta f + \frac{1}{m} R f)\|G\|^2$$

and the relations (3.5) and (3.7) give to

$$\frac{1}{2}\Delta\|G\|^2 - \|\nabla f\|^2 = G\Delta f + \frac{1}{m} R\|G\|^2.$$

Substituting these two equations into (3.6), we have

$$\overline{R} = n[(n-1)\|G\|^2 - (n-2) f\Delta f - \frac{n-1}{m} R f^2 - f^2 G\Delta f/\|G\|^2]. \tag{3.12}$$

If the function f is a special concircular scalar field on M_1, then we see from (3.11) that

$$\Delta f = -\frac{R}{m-1} f.$$

Substituting this equation into (3.12), we obtain

$$\overline{R} = n(n-1)(\|G\|^2 + \frac{1}{m(m-1)} R f^2).$$

Thus we can state

THEOREM 3.2. *Let $M = M_1 \times {}_f \overline{M}_2$ be an AEWP of an m (≥ 2)-dimensional space M_1 of constant curvature R and an n-dimensional Riemannian manifold $\overline{M}_2$. Suppose that M has harmonic curvature. Then the function f satisfies the equation*

$$n[(n-1)\|G\|^2 - (n-2) f\Delta f - \frac{n-1}{m} R f^2 - f^2\|G\|^{-2} G\Delta f] = \overline{R},$$

where $\overline{R}$ is the constant curvature of $\overline{M}_2$. If f is a special concircular scalar field, then it satisfies

$$n(n-1)(\|G\|^2 + \frac{1}{m(m-1)} Rf^2) = \overline{R}.$$

In the case where M_1 is a 1-dimensional, we shall denote the derivative with respect to the coordinate x^1 of M_1 by prime. Then it follows from (3.6) that

$$\overline{R} = n[(n-1)(f')^2 - (n-1)ff'' - f^2(f')^{-1}f'''].\tag{3.13}$$

We can easily see from (3.1) and (3.13) that M has harmonic curvature if and only if $\overline{M}_2$ is an Einstein manifold with constant scalar curvature $\overline{R}$ given by (3.13). Thus we can state

THEOREM 3.3. *Let $M = M_1 \times_f \overline{M}_2$ be an AEWP of a 1-dimensional manifold M_1 and an n-dimensional Riemannian manifold $\overline{M}_2$. Then M has harmonic curvature if and only if $\overline{M}_2$ is an Einstein manifold and the function f satisfies the ordinary differential equation*

$$n[(n-1)(f')^3 - (n-1)ff'f'' - f^2f'''] = \overline{R}f',\tag{3.14}$$

where $\overline{R}$ is the constant scalar curvature of $\overline{M}_2$.

REMARK 3.4. Under the assumptions of Theorem 3.3, it follows from (2.6) that

$$\begin{cases} D_1K_{11} = -nf^{-2}(ff''' - f'f''), \\ D_rK_{1q} = D_1K_{rq} = (ff''' - f'f'')\overline{g}_{rq}, \end{cases}\tag{3.15}$$

and otherwise vanished. Therefore we see from (3.15) that the Ricci tensor of M is not parallel in general

IV. The parallel Ricci tensors

In this paragraph, we shall deal with an AEWP $M = M_1 \times {}_f\overline{M}_2$ of two Riemannian manifolds M_1 and $\overline{M}_2$, and assume that the Ricci tensor K_{BA} of M is parallel, that is, $D_CK_{BA} = 0$. Then it follows from the first relation of (2.6) that

$$f^2\nabla_k R_{ji} = n(f\nabla_k\nabla_j f_i - f_k\nabla_j f_i).\tag{4.1}$$

Applying g^{ji} and g^{ki} to (4.1), and summing up the repeated indices, we have

$$f^2\nabla_j R = n(f\nabla_j\Delta f - f_j\Delta f) \tag{4.2}$$

and

$$f^2\nabla_j R = n(2f\Delta f_j - \nabla_j\|G\|^2) \tag{4.3}$$

respectively. Comparing (4.2) with (4.3), we obtain

$$f\nabla_j\Delta f - f_j\Delta f = 2f\Delta f_j - \nabla_j\|G\|^2. \tag{4.4}$$

Since the Ricci identity (3.4) implies

$$f^i R_{ji} = \Delta f_j - \nabla_j\Delta f, \tag{4.5}$$

the equation (4.4) reduces to

$$2ff^i R_{ji} = \Delta_j\|G\|^2 - f_j\Delta f - f\nabla_j\Delta f. \tag{4.6}$$

From the third relation of (2.6) and the above (4.6), we obtain

$$f_j\overline{R}_{rq} = [\frac{1}{2}ff_j\Delta f + (n-1)\|G\|^2 f_j - \frac{1}{2}f^2\nabla_j\Delta f - \frac{1}{2}(n-1)f\nabla_j\|G\|^2]\overline{g}_{rq}, \tag{4.7}$$

which follows from the fourth of (2.6).

If the function f has non-empty zero-level surface, then we see from (4.1) and (4.7) that

$$\nabla_j f_i = 0 \quad \text{and} \quad \overline{R}_{rq} = (n-1)\|G\|^2\overline{g}_{rq}.$$

Conversely if the relations (4.1) and (4.7) are satisfied on M, we easily see from (2.6) that the Ricci tensor of M is parallel. Thus we can state

THEOREM 4.1. *Let $M = M_1 \times {}_f\overline{M}_2$ be an AEWP of two Riemannian manifolds M_1 and $\overline{M}_2$ of dimensions m and n respectively. Then the Ricci tensor of M is parallel if and only if (1) the covariant derivative of the Ricci tensor of M_1 satisfies*

$$\nabla_k R_{ji} = nf^{-2}(f\nabla_k\nabla_j f_i - f_k\nabla_j f_i),$$

and (2) the Ricci tensor of $\overline{M}_2$ does

$$2f_j\overline{R}_{rq} = [ff_j\Delta f + 2(n-1)\|G\|^2 f_j - f^2\nabla_j\Delta f - (n-1)f\nabla_j\|G\|^2]\overline{g}_{rq},$$

provided f has empty zero-level surface.

In this case where f has non-empty zero-level surface,

$$\nabla_j f_i = 0 \quad \text{and} \quad \overline{R}_{rq} = (n-1)\|G\|^2\,\overline{g}_{rq}$$

are satisfied on M.

We assume that M_1 is an m (≥ 2)-dimensional space of constant curvature. Then we see from (3.7) and (4.2) that

$$f\nabla_j\Delta f = f_j\Delta f. \tag{4.8}$$

It follows from (4.3), (4.5) and (4.7) that

$$\nabla_j\|G\|^2 = 2(\Delta f + \frac{1}{m}Rf)f_j, \tag{4.9}$$

where R is the constant scalar curvature of M_1. Substituting (4.8) and (4.9) into (4.7), we obtain

$$\overline{R}_{rq} = (n-1)(\|G\|^2 - f\Delta f - \frac{1}{m}Rf^2)\overline{g}_{rq}. \tag{4.10}$$

Therefore the following is immediate from Theorem 4.1

THEOREM 4.2. *Let $M = M_1 \times {}_f\overline{M}_2$ be an AEWP of an m ($\geq$ 2)-dimensional space M_1 of constant curvature and an n-dimensional Riemannian manifold $\overline{M}_2$. Then the Ricci tensor of M is parallel if and only if (1) the function f satisfies*

$$f\nabla_k\nabla_j f_i = f_k\nabla_j f_i,$$

and (2) $\overline{M}_2$ is an Einstein manifold with the constant scalar curvature

$$\overline{R} = n(n-1)(\|G\|^2 - f\Delta f \frac{1}{m} R f^2),$$

where R is the constant scalar curvature of M_1.

If M_1 is a 1-dimensional manifold, and if we denote the derivative with respect to the coordinate x^1 of M_1 by prime, then it follows from (4.1) that

$$ff''' = f'f''. \tag{4.11}$$

Taking account of (4.11), the relation (4.7) reduces to

$$\overline{R}_{rq} = (n-1)(f'^2 - ff'')\overline{g}_{rq}.$$

Thus the following is also immediate from Theorem 4.1.

THEOREM 4.3. *Let $M = M_1 \times_f \overline{M}_2$ be an AEWP of a 1-dimensional space M_1 and an n-dimensional Riemannian manifold $\overline{M}_2$. Then the Ricci tensor of M is parallel if and only if $\overline{M}_2$ is Einstein and the ordinary differential equation*

$$n(n-1)(f'^2 - ff'') = \overline{R}$$

is satisfied, where $\overline{R}$ is the constant scalar curvature of $\overline{M}_2$.

V. The Bourguignon's conjucture

In this paragraph, we shall give a negative answer for the so-called Bourguignon's conjucture. The conjucture suggested as "the Ricci tensor of a compact Riemannian manifold with harmonic curvature must be parallel", and A. Derdzinski gave an example as for a negative answer of it in [3].

Let $M = M_1 \times_f \overline{M}_2$ be an AEWP of a 1-dimensional manifold M_1 and an n-dimensional Riemannian manifold $\overline{M}_2$, and denote the coordinate x^1 of M_1 by t. As stated in Theorem 3.3 and Remark 3.4 in paragraph III, M has harmonic curvature if and only if $\overline{M}_2$ is Einstein, the function $f(t)$ satisfies

$$n[(n-1)f'^3 - (n-2)ff'f'' - f^2f'''] = \overline{R}f'. \tag{5.1}$$

The non-vanishing derivative of components of the Ricci tensor of M are given by

$$\begin{cases} D_1K_{11} = nf^{-2}(f'f'' - ff'''), \\ D_rK_{1q} = D_1K_{rq} = (ff''' - f'f'')\overline{g}_{rq}. \end{cases} \tag{5.2}$$

As in Theorem 4.3, the Ricci tensor of M is parallel if and only if $\overline{M}_2$ is Einstein and the function $f(t)$ satisfies

$$n(n-1)(f'^2 - ff'') = \overline{R}. \tag{5.3}$$

Differentiating (5.3) with respect to t, we have

$$f'f'' = ff''',$$

and hence the equation (5.3) is rewritten as

$$n(n-1)(f'^2 - cf^2) = \overline{R}, \tag{5.4}$$

where c is a constant on M.

We put $a = [\overline{R}/n(n-1)]^{1/2}$ and $c = -b^2, 0, b^2$ according to the sign of c, where b is a positive constant. Then, by a suitable choice of the first coordinate t of the separate coordinate system $(t, x^2, \cdots, x^{n+1})$ of M, the solution of the equation (5.4) is given by

$$f(t) = \begin{cases} at & \text{for } c = 0, \\ \exp bt & \text{for } c = b^2,\ \overline{R} = 0, \\ (a/b)\sinh bt & \text{for } c = b^2,\ \overline{R} > 0, \\ -(a/b)\cosh bt & \text{for } c = b^2,\ \overline{R} < 0, \\ (a/b)\cos bt & \text{for } c = -b^2,\ \overline{R} > 0. \end{cases} \tag{5.5}$$

It is easily seen from (5.2) that the Ricci tensor of M is parallel if and only if the function $f(t)$ satisfies (5.4), that is, it is equal to a function in (5.5). Therefore if $f(t)$ satisfies the equation (5.1) but does not satisfy (5.4), or even if it is equal to a function of the same type in (5.5) with some different coefficients to the constant a, then M has harmonic curvature but the Ricci tensor of M is not parallel. Summing up these results, we can state

THEOREM 5.1. *Let $M = M_1 \times_f \overline{M}_2$ be an AEWP of a 1-dimensional manifold M_1 and an n-dimensional Riemannian manifold $\overline{M}_2$. Then M has harmonic curvature and non-parallel Ricci tensor if and only if (1) $\overline{M}_2$ is Einstein, and (2) the function f is equal to a solution of the ordinary differential equation*

$$n[(n-1)f'^3 - (n-2)ff'f'' - f^2f'''] = \overline{R}f',$$

which does not equal to a solution of the equation

$$n(n-1)(f'^2 - cf^2) = \overline{R},$$

c and $\overline{R}$ being a constant and constant scalar curvature of $\overline{M}_2$ respectively.

By virtue of Theorem 5.1, we may construct many compact AEWP with harmonic curvature and non-parallel Ricci tensor. For example, if we choose $\overline{M}_2$ as a compact Einstein manifold and the function f as a solution described in Theorem 5.1, then the Riemannian manifolds $I \times_f \overline{M}_2$ and $S \times_f \overline{M}_2$ are compact AEWP's and have the properties mentioned above, where I and S indicate a closed interval and a circle respectively.

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