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SOCLE OF C–M LOCAL RINGS AND STRONG F–REGULARITY

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0. Introduction

In [1], M. Hochster proved that if (R, m) is a regular local ring, $x_1, \cdots, x_d$ is a regular system of parameters for R and if $R \subset S$, where S is a module-finite R-algebra, then R is a direct summand of S if and only if for every integer k, $(x_1 \cdots x_d)^k \notin (x_1^{k+1}, \ldots, x_d^{k+1})S$. In Theorem 3.3, we generalize the above theorem in the case when R is a complete and reduced Noetherian Cohen-Macaulay (for short, C–M) local ring and u_k generates the minimum nonzero ideal of R/I_k for every k. And in Proposition 2.7, we shall prove that if the elements $y_1, \ldots, y_n \in R$ generate the socle of R/J_1, then $y_1 u_t, \ldots, y_n u_t$ generate the socle of R/J_{t+1}, where $J_{t+1} = (x_1^{t+1}, \cdots, x_d^{t+1})$ and $u_t = x_1^t \cdots x_d^t$. Using the Proposition 2.7, we prove that if (R, m) is F-rational and Gorenstein, then R is strongly F-regular in Theorem 3.5.

1. Preliminaries and definitions

Throughout this paper, all rings are Noetherian and commutative with 1 and positive characteristic p. We introduce the notion of the tight closure of an ideal I in R. Let $R^0 = R - \cup\{P : P \text{ is a minimal prime of } R\}$.

DEFINITION 1.1. Let $I \subseteq R$ be an ideal. If R is a ring with characteristic $p > 0$, we say that $x \in R$ is in the *tight closure*, I^*, of I, if there exists $c \in R^0$ such that for all $e \gg 0$, $cx^{p^e} \in I^{[p^e]}$, where $I^{[q]} = (i^q : i \in I)$ when $q = p^e$. If $I = I^*$, we say that I is tightly closed.

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REMARK 1.2. (a) If R is regular, then $I = I^*$ for all I. (ref., [3], [4], [5], [6]).

(b) A Gorenstein local ring has the property that $I = I^*$ for all I if and only if the ideal generated by a single system of parameters is tightly closed. (ref. [3].)

DEFINITION 1.3. R is ***weakly F-regular*** if every ideal in R is tightly closed. R is ***F-regular*** if R_p is weakly F-regular for every $p \in \mathrm{Spec}(R)$.

DEFINITION 1.4. R is ***F-rational*** if every ideal generated by a system of parameters is tightly closed.

2. Socle of a C-M local rings

DEFINITION 2.1. Let R be a reduced Noetherian ring of positive prime characteristic p such that $R^{\frac{1}{p}}$ is module-finite over R. (Of course, $R^{\frac{1}{q}}$ is then module-finite over R for all $q = p^e$). We say that R as above is ***strongly F-regular*** if for every $c \in R^o$, there exists $q = p^e$ such that the R-linear map $R \longrightarrow R^{\frac{1}{q}}$ which sends 1 to $c^{\frac{1}{q}}$ splits as a map of R-modules, i.e., if and only if $Rc^{\frac{1}{q}} \subset R^{\frac{1}{q}}$ splits over R.

REMARK 2.2 ([5] 3.1 THEOREM (D)). If R is strongly F-regular, then R is F-regular.

DEFINITION-PROPOSITION 2.3 ([2] (1.1) DEFINITION-PROPOSITION). A Noetherian local ring (R, m) is ***approximately Gorenstein*** if it satisfies either of the following equivalent conditions:

(a) *For every integer $N > 0$ there is an ideal $I \subset m^N$ such that R/I is Gorenstein.*

(b) For every integer $N > 0$ there is an m-primary irreducible ideal $I \subset m^N$.

THEOREM 2.4 ([2] (1.6) THEOREM). *If (R, m) is a complete and reduced (or even an excellent and reduced) local ring with $dim R \geq 1$. Then R is approximately Gorenstein if and only if the following two conditions hold:*

(a) $m \notin \mathrm{Ass}(R)$, *i.e.,* $\mathrm{depth} R \geq 1$.

(b) *If $P \in \mathrm{Ass}(R)$ and $\dim R/P = 1$, then $(R/P) \oplus (R/P)$ is not embeddable in R.*

DEFINITION 2.5. If (R, m) is a Noetherian local ring and q is an m-primary ideal, then the R-module R/q is of finite length. The *socle* of R/q is the set of all residue classes $\bar{r} \in R/q$ that are annihilated by m :

$$\mathcal{S}(R/q) := \{\bar{r} \in R/q | m \cdot \bar{r} = 0\}.$$

REMARK 2.6. (a) ([7] Proposition VI.3.17). Let (R, m) be a C–M local ring and $a_1, \cdots, a_d$ be a system of parameters for R. Then the number

$$r := \dim_{R/m}(\mathcal{S}(R/(a_1, \cdots, a_d)))$$

is independent of the choice of the system of parameters $a_1, \cdots, a_d$.

(b) ([7] Definition VI.3.18). If the number $r = 1$ in (a), then (R, m) is a Gorenstein local ring.

PROPOSITION 2.7. *Let (R, m) be a C–M local ring with dimension d and let $J = J_1 = (x_1, \cdots, x_d)$ be an ideal generated by a system of parameters for R. If the elements $y_1, \cdots, y_n \in R$ generate the socle of R/J_1, then $y_1 u_t, \cdots, y_n u_t$ generate the socle of R/J_{t+1}, where $J_{t+1} = (x_1^{t+1}, \cdots, x_d^{t+1})$ and $u_t = x_1^t \cdots x_d^t$.*

Proof. **Case 1.** $\dim R = 1$.

$$\mu_{x_1} : \mathcal{S}(R/Rx_1) \longrightarrow \mathcal{S}(R/Rx_1^2)$$

is an isomorphism, where $\mathcal{S}(*)$ is the socle of $*$ and μ_{x_1} is the multiplication by x_1.

i) μ_{x_1} is injective; for

$$\begin{aligned} mr \subset x_1 R \text{ and } rx_1 \in (x_1^2) &\Longrightarrow rx_1 = x_1^2 t, \text{ for some } t \in R \\ &\Longrightarrow r = x_1 t. \end{aligned}$$

It follows that μ_{x_1} is injective.

ii) μ_{x_1} is surjective; for

$$\begin{aligned} mr \subset (x_1^2) \text{ and } r \notin (x_1^2) &\Longrightarrow x_1 r \in mr \subset (x_1^2) \\ &\Longrightarrow r = x_1 t \text{ for some } t \in R; \\ &\qquad \text{for } rm = x_1 tm \subset (x_1^2) \\ &\Longrightarrow tm \subset (x_1). \end{aligned}$$

This means that $t \in \mathcal{S}(R/Rx_1)$ and $tx_1 = r \Longrightarrow \mu_{x_1}$ is surjective. By i) and ii) μ_{x_1} is an isomorphism. Inductively, we can prove that

$$\mathcal{S}(R/x_1R) \simeq \mathcal{S}(R/x_1^{t+1}R) \quad \text{and} \quad \mathcal{S}(R/x_1^{t+1}R) = <\overline{yx_1^t}> .$$

Case 2. $\dim R = d \geq 2$.

Let $\bar{R} = R/Rx_d$ and let $\tilde{R} = R/(x_1^{t+1}, \cdots, x_{d-1}^{t+1})R$. Then $\bar{R}$ and $\tilde{R}$ are also C–M and

$$\begin{array}{ccc}
\mathcal{S}(R/(x_1,\cdots,x_d)) & \xrightarrow{=} & \mathcal{S}(\bar{R}/(x_1,\cdots,x_{d-1})\bar{R}) \\
 & & \downarrow {\scriptstyle f}\,\simeq \\
\mathcal{S}(R/(x_1^{t+1},\cdots,x_{d-1}^{t+1},x_d)) & \xleftarrow{=} & \mathcal{S}(\bar{R}/(x_1^{t+1},\cdots,x_{d-1}^{t+1})\bar{R}) \\
\downarrow = & & \\
\mathcal{S}(\tilde{R}/(x_d)\tilde{R}) & \xrightarrow[\simeq]{g} & \mathcal{S}(\tilde{R}/(x_d^{t+1})\tilde{R}) \\
 & & \downarrow = \\
 & & \mathcal{S}(R/(x_1^{t+1},\cdots,x_d^{t+1})),
\end{array}$$

where for every $\bar{r} \in \mathcal{S}(\bar{R}/(x_1,\cdots,x_{d-1})\bar{R}), f(\bar{r}) = \overline{rx_1^t\cdots x_d^t} \in \mathcal{S}(\bar{R}/(x_1^{t+1},\cdots,x_{d-1}^{t+1})\bar{R})$ and for every $\bar{r}' \in \mathcal{S}(\tilde{R}/(x_d)\tilde{R}), g(\bar{r}') = \overline{r'x_d^t} \in \mathcal{S}(\tilde{R}/(x_d^{t+1})\tilde{R})$. Hence $\mathcal{S}(R/J) \longrightarrow \mathcal{S}(R/J_{t+1})$ is an isomorphism. It follows that if $\mathcal{S}(R/J_1) = <\bar{y}_1,\cdots,\bar{y}_n>$, then

$$\mathcal{S}(R/(x_1^{t+1},\cdots,x_d^{t+1})) = <\overline{y_1u_t},\cdots,\overline{y_nu_t}> .$$

REMARK 2.8. Let (R, m) be a Gorenstein local ring with dimension d and let $J = J_1 = (x_1,\cdots,x_d)$ be an ideal generated by a system of parameters for R. If an element $y \in R$ generates the socle of R/J_1, then yu_t generates the socle of R/J_{t+1} by Proposition 2.7, where $J_{t+1} = (x_1^{t+1},\cdots,x_d^{t+1})$ and $u_t = x_1^t\cdots x_d^t$.

3. Contracted ideals of complete and reduced local rings and Strong F-regularity

LEMMA 3.1 ([1] LEMMA 1). *Let $R \subset S$ be rings and assume that S is finitely presented as an R-module. Then R is a direct summand of S if and only if for each maximal ideal m of R, R_m is a direct summand of S_m.*

Moreover, if T is a faithfully flat R-algebra, then R is a direct summand of S if and only if $T = R \otimes_R T$ is a direct summand of $S \otimes_R T$.

PROPOSITION 3.2 ([7] PROPOSITION 10). *If $R \subset S$ are rings such that R is a direct summand of S, then for each ideal I of R, $IS \cap R = I$.*

Throughout the rest of this section (R, m) denotes a Noetherian C–M local ring of positive prime characteristic p such that $R^{\frac{1}{p}}$ is module-finite over R unless otherwise specified. Let $\dim R = d$ and $x_1, x_2, ..., x_d$ be a system of parameters for R. We denote that $u_k = x_1^k \cdots x_d^k$.

By Theorem 2.4, if (R, m) is a complete and reduced (or even an excellent) local ring, then there must exist a sequence $\{I_k\}$ of irreducible m-primary ideals cofinal with the powers of m.

THEOREM 3.3. *Let (R, m) be a complete and reduced Noetherian C–M local ring with 1. Let $x_1, \cdots, x_d$ be a system of parameters for R and I_k, u_k be as before. Assume that $u_k \in R$ generate the minimum nonzero ideal of R/I_{k+1} modulo I_{k+1} for every k. Let $R \subset S$, where S is a module finite R-algebra. Then R is a direct summand of S if and only if for every integer $k > 0$, $u_k \notin I_{k+1}S$.*

Proof. Let $R_k = R/I_k$. Then $R_k = R/I_k$ is Gorenstein and zero-dimensional: let $m_k = m/I_k$. Then for every nonzero ideal J/I_{k+1} in R_{k+1}, $u_k \in J/I_{k+1}$ by given hypothesis. It follows that every ideal of R strictly larger than I_{k+1} contains u_k.

Now, if R is a direct summand of S, then every ideal of R will be contracted by Proposition 3.2. Since $u_k \notin I_{k+1}R$, we will have $u_k \notin I_{k+1}S$, which is precisely the condition asserted in Theorem 3.3.

To prove the converse, assume that $u_k \notin I_{k+1}S$, for all k. Since the ideals I_k are cofinal with the powers of m and R is complete,

$$(*) \qquad \mathrm{Hom}_R(S, R) = \varprojlim_k \mathrm{Hom}_R(S_k, R_k)$$

where $S_k = S/I_kS$. We also note the isomorphism

$$\mathrm{Hom}_R(S_k, R_k) \simeq \mathrm{Hom}_{R_k}(S_k, R_k).$$

Now, $I_kS \cap R = I_k$, because if $J = I_kS \cap R$, $I_k \subset J$, then $u_k \in J$, this contradicts our hypothesis. Hence, the inclusion $R \hookrightarrow S$ induces an inclusion $R_k \hookrightarrow S_k$ for each k. Since R_k is a zero-dimensional Gorenstein local ring, it is injective as an R_k-module, and it follows that for each k, the inclusion $R_k \hookrightarrow S_k$ splits, i.e., R_k is a direct summand of S_k. For each k, let $h_k : \mathrm{Hom}_R(S_k, R_k) \longrightarrow R_k$ by $h(\phi) = \phi(1_k)$ where $1_k \in R_k \subset S_k$ in the image of $1 \in R \subset S$, and let $h : \mathrm{Hom}_R(S, R) \longrightarrow R$ by $h(\phi) = \phi(1)$. Then $H_k = h_k^{-1}(1)$ (respectively, $H = h^{-1}(1)$) is the set of splitting of $R_k \longrightarrow S_k$ (respectively, $R \longrightarrow S$) and the inverse limit relation $(*)$ induces

$$(**) \qquad H = \varprojlim_k H_k.$$

Here H_k (respectively, H) is a coset (or translate) of a submodule of $\mathrm{Hom}_R(S, R_k)$ (respectively, $\mathrm{Hom}_R(S, R)$) and the maps are restricted module homomorphisms. All we need to show to complete the proof is that $H \neq \emptyset$. Since each R_k is a direct summand of S_k, each $H_k \neq \emptyset$. But an inverse limit of nonempty cosets in Artinian modules is nonempty. To see this, note that for each k the decreasing sequence of nonempty subcosets $\mathrm{Im}(H_{i+k} \longrightarrow H_k)$ of H_k stabilizes, since their lengths must stabilize. Denote this subcoset of H_k by E_k. Then the E_k form a subsystem of nonempty subcosets and surjective maps so that

$$\emptyset \neq \varprojlim_k E_k \subset H.$$

This completes the proof.

COROLLARY 3.4. *Let (R, m) be a Gorenstein local ring. Let $x_1, \cdots, x_d$ be a system of parameters for R. Assume that the image of y in R/I_1 generates the socle $S(R/I_1)$ of R/I_1, where $I_k = (x_1^k, \cdots, x_d^k)$. Let $R \subset S$, where S is a module finite R-algebra. Then R is a direct summand of S if and only if for every integer $k > 0$,*

$$yu_k \notin I_{k+1}S.$$

Proof. Since R is a Gorenstein local ring, $\{I_k\}$ is a sequence of irreducible m-primary ideals cofinal with the powers of m. Also, $R_k = R/I_k$ is a zero-dimensional Gorenstein local ring. Let $m_k = m/I_k$. Then $\text{Ann}_{R_k} m_k$ is isomorphic to a single copy of R/m, and every nonzero ideal of R_k contains $\text{Ann}_{R_k} m_k$. It is quite easy to see, in fact, that $\text{Ann}_{R_k} m_k$ is generated by the residue class modulo I_k of $yx_1^{k-1} \cdots x_d^{k-1}$ by Remark 2.8. It follows that every ideal of R strictly larger than I_k contains $yx_1^{k-1} \cdots x_d^{k-1}$.

The rest of this proof is precisely the same as the proof of Theorem 3.3.

THEOREM 3.5. *If (R, m) is F-rational and Gorenstein, then R is strongly F-regular.*

Proof. Let $x_1, \cdots x_d$ be a system of parameters for R. Then, since R is Gorenstein, $I_k = \{(x_1^k, \cdots, x_d^k)\}$ is a sequence of irreducible m-primary ideals cofinal with the powers of m. Let the socle $S(R/I_1)$ be generated by $y \in R$ and $u_k = x_1^k \cdots x_d^k$.

For every $c \in R^0$, if $yu_k c^{\frac{1}{q}} \in I_{k+1} R^{\frac{1}{q}}$ for some integer $q = p^e, k > 0$, then $yc^{\frac{1}{q}} \in I_1 R^{\frac{1}{q}}$ since $x_1^{kq}, \cdots, x_d^{kq}$ is also a regular sequence. But if $yc^{\frac{1}{q}} \in I_1 R^{\frac{1}{q}}$ for all $q > 0$, then $y \in I^* = I$, a contradiction. Thus $yc^{\frac{1}{q}} \notin I_1 R^{\frac{1}{q}}$ for some $q > 0$. Hence $yu_k c^{\frac{1}{q}} \notin I_{k+1} R^{\frac{1}{q}}$ for every $k > 0$. It follows that $Rc^{\frac{1}{q}} \subset R^{\frac{1}{q}}$ splits over R by Corollary 3.4. Therefore, R is strongly F-regular.

REMARK 3.6. It is not necessary to assume that S is an R-algebra. Assume that R,u_k and I_k are as in Theorem 3.3. Assume that $u_k \in R$ generate the minimum nonzero ideal of R/I_k modulo I_k for every k. Let E be a finitely generated R-module and $e \in E$ be such that $\text{Ann}(e) = (0)$. *Then Re is a direct summand of E if and only if for every integer $k > 0$, $u_k e \notin I_{k+1} E$.* The proof of this result follows precisely the lines of the proof Theorem 3.3. Hence we can obtain the following Theorem 3.7.

THEOREM 3.7. *Let (R, m) be a complete and reduced Noetherian C–M local ring of positive prime characteristic p with 1, let $x_1, \cdots, x_d$, u_k and I_k be as in Theorem 3.3. Assume that $u_k \in R$ generate the minimum nonzero ideal of R/I_{k+1} modulo I_{k+1} for every k. Let E be a finitely generated R-module and $e \in E$ be such that* $\text{Ann}(e) = (0)$.

Then Re is a direct summand of E if and only if for every integer $k > 0$, $u_k e \notin I_{k+1}E$.

COROLLARY 3.8. *Let $R, x_1, \cdots, x_d, u_k$ and I_k be as in Theorem 3.3 and $R^{\frac{1}{p}}$ be module-finite. Assume that $u_k \in R$ generate the minimum nonzero ideal of R/I_{k+1} modulo I_{k+1} for every k. Then for every $c \in R^0$, $\mathrm{Ann}(c) = 0$. Hence, for all $q = p^e > 0$, $Rc^{\frac{1}{q}}$ is a direct summand of $R^{\frac{1}{q}}$ if and only if $u_k c^{\frac{1}{q}} \notin I_{k+1}R^{\frac{1}{q}}$ for every integer $k > 0$.*

LEMMA 3.9 ([5] (3.1) THEOREM). *Let R be a Noetherian ring of positive characteristic p such that $R^{\frac{1}{p}}$ is module finite over R. Then R is storngly F-regular if and only if R_P is strongly F-regular for every prime (respectively, for every maximal) ideal P of R.*

In [5], M. Hochster and C. Huneke prove that if R is regular and $R^{\frac{1}{p}}$ is module-finite over R, then R is strongly F-regular. We will give another proof of it in the following Theorem 3.10.

THEOREM 3.10. *If R is regular and $R^{\frac{1}{p}}$ is module-finite over R, then R is strongly F-regular.*

Proof. To prove strongly F-regularity, it suffices to do so locally, by Lemma 3.9. We may assume that (R, m) is local. Let $x_1, \ldots, x_d$ be a regular system of parameters for R and let $I_k = (x_1^k, \cdots, x_d^k)$. Then for every $c \in R^0$, if $u_k c^{\frac{1}{q}} \in I_{k+1}R^{\frac{1}{q}}$ for some integer $q = p^e, k > 0$, then $c^{\frac{1}{q}} \in I_1 R^{\frac{1}{q}}$ since $x_1^{kq}, \ldots, x_d^{kq}$ is also a regular sequence. But if $c^{\frac{1}{q}} \in I_1 R^{\frac{1}{q}}$ for all $q > 0$, then $c \in I_1^{[q]}$ for all $q > 0$, i.e., $c \in \cap I_1^{[q]} \subset \cap I_1^q = 0$ by the Nakayama's Lemma and the Krull intersection theorem, a contradiction. Thus $c^{\frac{1}{q}} \notin I_1 R^{\frac{1}{q}}$ for some $q > 0$. Hence $u_k c^{\frac{1}{q}} \notin I_{k+1}R^{\frac{1}{q}}$ for every $k > 0$. It follows that $Rc^{\frac{1}{q}} \subset R^{\frac{1}{q}}$ splits over R by Theorem 3.3. Therefore, R is strongly F-regular.

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